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New Aspects of Heterotic–F Theory Duality

Hirotaka Hayashi[1], Radu Tatar[2],
Yukinobu Toda[3], Taizan Watari[1,3] and Masahito Yamazaki[1]

[1] *Department of Physics, University of Tokyo, Tokyo 113-0033, Japan*

[2] *Division of Theoretical Physics, Department of Mathematical Sciences, The University of Liverpool, Liverpool, L69 3BX, England, U.K.*

[3] *Institute for the Physics and Mathematics of the Universe, University of Tokyo, Kashiwano-ha 5-1-5, 277-8592, Japan*

Abstract

In order to understand both up-type and down-type Yukawa couplings, F-theory is a better framework than the perturbative Type IIB string theory. The duality between the Heterotic and F-theory is a powerful tool in gaining more insights into F-theory description of low-energy chiral multiplets. Because chiral multiplets from bundles $\wedge^2 V$ and $\wedge^2 V^\times$ as well as those from a bundle V are all involved in Yukawa couplings in Heterotic compactification, we need to translate descriptions of all those kinds of matter multiplets into F-theory language through the duality. We find that chiral matter multiplets in F-theory are global holomorphic sections of line bundles on what we call covering matter curves. The covering matter curves are formulated in Heterotic theory in association with normalization of spectral surface, while they are where $M2$-branes wrapped on a vanishing two-cycle propagate in F-theory. Chirality formulae are given purely in terms of primitive four-form flux. In order to complete the translation, the dictionary of the Heterotic–F theory duality has to be refined in some aspects. A precise map of spectral surface and complex structure moduli is obtained, and with the map, we find that divisors specifying the line bundles correspond precisely to codimension-3 singularities in F-theory.

Contents

1 Introduction

For a description of our real world, we need both up-type quark Yukawa couplings

$$\Delta W = \mathbf{10}^{ab}\ \mathbf{10}^{cd}\ H(\mathbf{5})^e\ \epsilon_{abcde} \tag{1}$$

and down-type quark (and charged lepton) Yukawa couplings

$$\Delta W = \bar{\mathbf{5}}_a\ \mathbf{10}^{ab}\ \bar{H}(\bar{\mathbf{5}})_b. \tag{2}$$

Here, we used a notation of effective theory with $\mathrm{SU}(5)_{\mathrm{GUT}}$ symmetry and $\mathcal{N} = 1$ supersymmetry. Perturbative super Yang–Mills interactions of open strings of Type IIA / IIB string theory may be able to give rise to the latter, but it is difficult to generated the up-type Yukawa couplings with $\mathrm{SU}(5)_{\mathrm{GUT}}$ indices contracted by an epsilon tensor.[1] Heterotic $E_8 \times E_8'$

[1]Theories without $\mathrm{SU}(5)_{\mathrm{GUT}}$ unification do not have this problem. It should be remembered, however, that Supersymmetric Standard Models without unification would need an extra explanation for apparent gauge coupling unification, and Pati–Salam type theories need some mechanism to make sure that quark doublets and lepton doublets have totally different electroweak mixing, although they belong to the same irreducible representation of the Pati–Salam gauge group.

string theory, G_2-holonomy compactification of 11-dimensional supergravity and F-theory compactification, however, are capable of generating both types of Yukawa couplings [1].

There are some motivations to develop theoretical tools to extract physics out of G_2-holonomy compactification of 11-dimensional supergravity or Calabi–Yau four-fold compactification of F-theory.

- Perturbative Heterotic $E_8 \times E_8'$ string theory predicts all of the GUT scale, the Planck scale and the value of unified gauge coupling constant [2], but not all of them turn out right within the perturbative regime. The dilaton expectation value should be in the strongly-coupled regime to fit the data. Certainly the Heterotic M-theory [3] can cover strong-coupling region of the moduli space of the Heterotic $E_8 \times E_8'$ string theory, but that is not the only possibility. F-theory, for example, also describes some parts of the strong coupling region of the moduli space of the Heterotic string theory [4].
- As long as we insist that both the up-type and down-type Yukawa couplings be obtained, we do not gain much freedom by replacing the Heterotic $E_8 \times E_8'$ string theory by G_2-holonomy compactification of eleven-dimensional supergravity or elliptic Calabi–Yau four-fold compactification of F-theory. In order to generate both types of Yukawa couplings, an underlying gauge symmetry of E_r $(r = 7, 8)$ is necessary [1]. Thus, there may be not much room to expect qualitatively different physics of quarks and leptons in vacua obtained by 11-dimensional supergravity or F-theory. Even at the qualitative level, however, one obtains greater freedom in constructing other sectors of the real world. It will be much easier in F-theory than in the Heterotic theory to construct models of gauge mediated supersymmetry breaking, for example.
- Flux compactification techniques [5] can be used to discuss observables in our visible sector in F-theory. In Heterotic string theory, it is really hard for now to discuss stabilization of vector bundle moduli. In Heterotic–F-theory duality, vector bundle moduli of Heterotic theory correspond to a part of complex structure moduli of F-theory [6, 7], and flux compactification of F-theory will stabilize such moduli as well as the rest of the complex structure moduli.

These motivations provide enough reasons to study F-theory, although they are not particularly in favor of F-theory over G_2-holonomy compactification of eleven-dimensional supergravity. Reference [1] discussed qualitative pattern of Yukawa matrices expected in G_2-holonomy compactification with unbroken $SU(5)_{\rm GUT}$ symmetry, and found that there is generically a problem in the texture of up-type Yukawa matrix. This situation adds a motivation to

develop a formulation to study Yukawa couplings in F-theory.

The most basic question in string phenomenology is who we are—what are quarks and leptons. These matter chiral multiplets in supersymmetric compactification are identified with independent elements of bundle-valued cohomology groups in the Heterotic, Type I and Type IIB string theory. The net chirality of matter multiplets in chiral representations is expressed in terms of topological numbers such as Euler characteristics of vector bundles or pairing of D-brane charges in K-theory. In Type IIA string compactification on a Calabi–Yau orientifold, we know that one chiral multiplet is localized at each D6–D6 intersection [8], and this local picture is extended to compactification of 11-dimensional supergravity on a G_2-holonomy "manifold" with A–D–E singularity [9]. We can provide satisfactory answers to the question above in all these theories. Surprisingly, though, such an effort to identify quarks and leptons in F-theory language went only halfway in 1990's, and almost came to a halt (at least to our knowledge), until the recent results of [10, 11].

The identification of quarks and leptons, or of chiral matter multiplets in general, has been such a challenging problem in F-theory, because an intrinsic formulation of F-theory has not been fully developed yet. The elementary degrees of freedom in F-theory can be described by (p,q) strings or $M2$-branes of 11-dimensional supergravity. It may be possible, to identify chiral matter multiplets on 3+1 dimensions with some of their fluctuation modes. In practice, however, it is extremely difficult to disentangle complicated geometry of triple intersection of (p,q) 7-branes, or to maintain distinction between left-handed and right-handed fermions in Calabi–Yau 4-fold compactification of 11-dimensional supergravity down to 2+1 dimensions. Instead, the duality between the Heterotic string and F-theory [4, 6, 7, 12, 13, 14, 15] will be the most powerful tool in studying F-theory.

This article is along the line of this approach; the Heterotic string theory and the Heterotic–F-theory duality are used to study F-theory. The Heterotic string theory compactified on an elliptically fibered Calabi–Yau 3-fold $\pi_Z : Z \to B_2$ is dual to F-theory compactified on an elliptically fibered Calabi–Yau 4-fold $\pi_X : X \to B_3$ whose base 3-fold B_3 is a $\mathbb{P}^1$ fibration over B_2. The various matter multiplets in low-energy effective theory are identified with $H^1(Z;\rho(V))$ in Heterotic string description, where $\rho(V)$ is a vector bundle V in representation ρ. Cohomology groups on a fibered space can be calculated first on the fiber geometry, and later on the base geometry; except for certain cases (which will be covered in section 3),

$$H^1(Z;\rho(V)) \simeq H^0(B_2; R^1\pi_Z{*}\rho(V)), \tag{3}$$

and the direct images $R^1\pi_{Z*}\rho(V)$ have their support only on curves in B_2. In the Heterotic–

F duality, these support curves correspond to 7-brane intersections, and the sheaves on the curves should be those on the 7-brane intersection curves. Chiral matter multiplets are identified with global holomorphic sections of such sheaves (except for certain cases). Thus, by using the Heterotic–F duality, we can obtain the sheaves whose sections are identified with quarks and leptons. Direct images $R^1\pi_{Z*}\rho(V)$ are, therefore, the information we would like to obtain from the Heterotic string theory.

Direct images of bundles in the fundamental representation $\rho(V) = V$ were obtained in 1990's [16, 17]. Those of bundles in the anti-symmetric representation $\rho(V) = \wedge^2 V$ have not been clearly described as sheaves so far in the last decade, apart from some developments in [18, 19] in the context of Heterotic theory compactification. Calculation of the direct images of $\wedge^2 V$, therefore, is one of the central themes in this article. This task is carried out in sections 4 and 5. This is by no means a minor problem. Both $\bar{\mathbf{5}}$ and $\bar{H}(\bar{\mathbf{5}})$ multiplets arise from $\wedge^2 V$ of an SU(5) bundle V, and $H(\mathbf{5})$ from $\wedge^2 V^\times$, where $V^\times$ is the dual bundle of V. Without understanding the geometry associated with $R^1\pi_{Z*} \wedge^2 V$ and $R^1\pi_{Z*} \wedge^2 V^\times$, there is no way to understand the Yukawa couplings of quarks and leptons in F-theory.

We introduce a new notion,[2] covering matter curve, (roughly speaking) in order to deal with singularities that appear along matter curves. The direct image $R^1\pi_{Z*}\wedge^2 V$ is represented as a pushforward of a locally free rank-1 sheaf $\widetilde{\mathcal{F}}_{\wedge^2 V}$ on the covering matter curve for all the cases we have study in section 5, i.e. for rank $V = 3, 4, 5, 6$. (We should also note here that a minor assumption is made on structure of $R^1\pi_{Z*} \wedge^2 V$ around a particular type of singularity for the rank $V = 4$ case.) Divisors determining the locally free rank-1 sheaves are determined in terms of data defining spectral surfaces.

In section 6, the description of the sheaves $R^1\pi_{Z*}\rho(V)$ are translated into language of F-theory. The dictionary between the Heterotic and F-theory quantities was almost established in 1990's, but it has to be refined in some aspects. Improvements include

- a precise map (240–251) between the moduli of spectral surface in Heterotic theory description and complex structure moduli in F-theory.
- a refinement of the correspondence between the discrete twisting data of vector bundles in Heterotic theory description and four-form fluxes in F-theory description. We are basically following the line of the idea laid out in [15, 10].

Using the precise map between the two moduli space, we find that most of the components of the divisors determining the sheaves $\mathcal{F}_{\rho(V)}$ (and all that we identified) correspond to

[2] Essentially the same object was already introduced in [19].

codimension-3 singularities in B_3.

The last section provides our conclusion and describes chiral matter multiplets in effective theory purely in a language of F-theory. All the results obtained in earlier sections are put together and provide a description of matter multiplets in a form that does not even assume an existence of Heterotic dual. Brief comments on Yukawa couplings are also found there.

Appendices A and B cover mathematical subjects that are necessary in sections 4 and 5, respectively.

2 Spectral Cover Construction and Direct Images

Heterotic string theory compactified on an elliptic Calabi–Yau 3-fold is dual to F-theory compactified on $K3$-fibered elliptic Calabi–Yau 4-fold. Once massless chiral multiplets are described in Heterotic string theory, the description can be passed on to F-theory, using the duality. In this section, we will review a powerful way to describe them that is known since late 1990's, mainly for the purpose of setting up notations used in this article.

Elliptic Fibration

Heterotic string theory has an F-theory dual description, if it is compactified on an elliptically fibered manifold. We consider an elliptic fibered Calabi–Yau 3-fold Z

$$\pi_Z : Z \to B_2 \tag{4}$$

over a base 2-fold, so that $\mathcal{N} = 1$ supersymmetry is left in low-energy effective theory below the Kaluza–Klein scale. An elliptic fibration Z over B_2 is given by a Weierstrass equation,

$$y^2 = x^3 + f_0 x + g_0. \tag{5}$$

Here, f_0 and g_0 are sections of line bundles $\mathcal{L}_H^{\otimes 4}$ and $\mathcal{L}_H^{\otimes 6}$ on B_2, respectively, and $\mathcal{L}_H \simeq \mathcal{O}(-K_{B_2})$ for Z to be a Calabi–Yau 3-fold. The coordinates (x, y) transform as sections of $\mathcal{L}_H^{\otimes 2}$ and $\mathcal{L}_H^{\otimes 3}$, respectively. The zero section $\sigma : B_2 \hookrightarrow Z$ maps B_2 to the locus of infinity points, $(x, y) = (\infty, \infty)$.

Spectral Cover Construction

Compactification of the Heterotic $E_8 \times E_8'$ string theory involves a pair of vector bundles (V_0, V_∞) on a Calabi–Yau 3-fold Z. Spectral cover construction [20, 12, 21] describes vector bundles on an elliptic fibered Calabi-Yau 3-fold Z. Let us consider a rank-N vector bundle V on Z. Spectral surface $C_V \in |N\sigma + \pi_Z^* \eta|$ is a smooth hypersurface of Z that is a degree N

cover over B_2, where η is a divisor on B_2. When a line bundle $\mathcal{N}_V$ on C_V is given, a rank-N vector bundle V is given by the Fourier–Mukai transform

$$V = p_{2*}(p_1^*(\mathcal{N}_V) \otimes \mathcal{P}_{B_2}), \tag{6}$$

where $p_{1,2}$ are maps associated with a fiber product

$$\begin{tikzcd} & C_V \times_{B_2} Z \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ C_V \arrow[dr, "\pi_C"'] & & Z \arrow[dl, "\pi_Z"] \\ & B_2 & \end{tikzcd} \tag{7}$$

$q := \pi_C \circ p_1 = \pi_Z \circ p_2$, and $\mathcal{P}_{B_2}$ is the Poincaré line bundle $\mathcal{O}_{C_V \times_{B_2} Z}(\Delta - \sigma_1 - \sigma_2 + q^* K_{B_2})$ with $\sigma_1 = \sigma \times Z, \sigma_2 = Z \times \sigma$ and Δ is a diagonal divisor of $Z \times Z$ restricted on $C_V \times_{B_2} Z$. The data $(C_V, \mathcal{N}_V)$, i.e. the spectral surface and a line bundle on it, determines a vector bundle V.

The characteristic classes of vector bundles constructed that way are expressed in terms of spectral data $(C_V, \mathcal{N}_V)$. The first Chern class of the vector bundle V is given by [12]

$$c_1(V) = \pi_Z^* \pi_{C*} \left(c_1(\mathcal{N}_V) - \frac{1}{2} r \right) \tag{8}$$

where $r := \omega_{C/B_2} := K_{C_V} - \pi_C^* K_{B_2}$ is the ramification divisor on C_V of $\pi_C : C_V \to B_2$, and $c_1(V)$ is a pullback of a 2-form on the base 2-fold B_2. With the notation

$$\gamma := c_1(\mathcal{N}_V) - \frac{1}{2} r, \tag{9}$$

we have

$$c_1(V) = \pi_Z^* \pi_{C*} \gamma. \tag{10}$$

We will sometimes use $c_1(V)$ in the sense of $\pi_{C*}\gamma$. The second Chern character is

$$\mathrm{ch}_2(V) = -\sigma \cdot \eta + \pi_Z^* \omega, \tag{11}$$

where ω is some 4-form on B_2 [12].

We do not restrict our attention to cases with vanishing first Chern class $c_1(V)$. By considering vector bundles V whose structure group is $\mathrm{U}(N)$, rather than $\mathrm{SU}(N)$, we will be able

to perform a consistency check in calculating $R^1\pi_{Z*}(\wedge^2 V)$ by examining $c_1(V)$ dependence. We maintain our discussion to be valid for $\mathrm{U}(N)$ bundles also because there are some phenomenological motivations to think of Heterotic string compactification with a bundle whose structure group is within $\mathrm{U}(N_1)\times\mathrm{U}(N_2)\subset\mathrm{SU}(5)$ [1].

We now present a few technical remarks about the nature of vector bundles given by spectral cover construction. Such bundles cannot be completely generic $\mathrm{U}(N)$ bundles. For example, the first Chern class $c_1(V)=c_1(\det V)$ is always given by a pullback of a 2-form on B_2 to Z (see (10)). In other words, the first Chern class of $\det V$ is trivial in the fiber direction. This is not a serious limitation when we are analysing Heterotic compactification in an attempt to understand F-theory better. In Heterotic string compactification, vector bundles have to be stable, and the stability condition (Donaldson–Uhlenbeck–Yau equation) is

$$\int_Z c_1(V)\wedge J\wedge J=0 \tag{12}$$

at tree level, where J is the Kähler form of Z. When the T^2-fiber is small, description in the Heterotic theory becomes less reliable, but a dual F-theory description becomes better. This is the situation we are interested in. In such a limit, the size of T^2-fiber becomes much smaller than that of the base, and the dominant contribution of (12) is from two J's in the two directions along B_2, and $c_1(V)$ in the fiber direction. Thus, the sole dominant contribution has to vanish, and hence stable vector bundles should not have non-vanishing $c_1(V)$ along the fiber direction.[3] Spectral cover construction, therefore, is fine for our purpose in this article, although it cannot describe a bundle with a non-vanishing first Chern class along the T^2-fiber direction.

For $\mathrm{U}(N)$ bundles given by spectral cover construction, $\det V$ are actually trivial along the elliptic fiber direction, not just degree zero. The spectral surface $C_V\hookrightarrow Z$ is (on a local patch of B_2) defined by the zero locus of an equation

$$s=a_0+a_2x+a_3y+a_4x^2+a_5xy+\cdots+a_N\left(x^{N/2}\text{ or }x^{(N-3)/2}y\right)=0, \tag{13}$$

where a_r are sections of $\mathcal{O}(\eta)\otimes\mathcal{L}_H^{\otimes(-r)}\simeq\mathcal{O}(rK_{B_2}+\eta)$ on B_2. The last term is $x^{N/2}$ or $x^{(N-3)/2}y$ depending on whether N is even or odd. On a given fiber $E_b:=\pi_Z^{-1}(b)$, s determines an elliptic function, with N zero points $\{p_i\}_{i=1,\cdots,N}$ (for $\mathrm{U}(N)$ bundles) and a rank-N pole at e_0, zero section σ on E_b. Since the group-law sum of the zero points of an elliptic function is the same as that of the poles,

$$\boxplus_i\, p_i=e_0, \tag{14}$$

[3] As long as $\int_{B_2}\pi_{C*}\gamma\wedge J=0$ is satisfied on B_2, vector bundles with non-vanishing $c_1(V)$ can be stable.

structure group of V	SU(2)	SU(3)	SU(4)	$\mathrm{SU}(5)_{\mathrm{bdl}}$	SU(6)
unbroken symmetry H	E_7	E_6	SO(10)	$\mathrm{SU}(5)_{\mathrm{GUT}}$	$\mathrm{SU}(3)\times\mathrm{SU}(2)$
from V	$\mathbf{56}$	$\mathbf{27}$	$\mathbf{16}$	$\mathbf{10}$	$(\bar{\mathbf{3}},\mathbf{2})$
from $V^\times$	(vct.-like)	$\overline{\mathbf{27}}$	$\overline{\mathbf{16}}$	$\overline{\mathbf{10}}$	$(\mathbf{3},\mathbf{2})$
from $\wedge^2 V$	—	—	$\mathbf{10}$	$\bar{\mathbf{5}}$	$(\mathbf{3},\mathbf{1})$
from $\wedge^2 V^\times$	—	—	(vct.-like)	$\mathbf{5}$	$(\bar{\mathbf{3}},\mathbf{1})$
from $\wedge^3 V$	—	—	—	—	$(\mathbf{1},\mathbf{2})$
from $\pi_Z^* E$	**adj.**	**adj.**	**adj.**	**adj.**	**adj.**

Table 1: If an SU(N) vector bundle V is turned on within an E_8 gauge group, E_8 symmetry is broken down to H, and chiral matter multiplets come out from various irreducible components of E_8-**adj**. decomposed under $\mathrm{SU}(N)\times H$. Irreducible components $(\rho(V), repr.)$ are denoted by *repr.* in this table. "(vct.-like)" in this table indicates that a given $(\rho(V)^\times, repr.)$ is the same as $(\rho(V), repr.)$ and self Hermitian conjugate in E_8 adjoint representation. The symmetry H may be further broken down by turning on a bundle E on the base manifold B_2. The structure group of E can be chosen, for example, in the $\mathrm{U}(1)_\chi$ direction in $H = \mathrm{SO}(10)$ so that the symmetry is broken down to $\mathrm{SU}(5)_{\mathrm{GUT}}$, or in the $\mathrm{U}(1)_Y$ direction in $H = \mathrm{SU}(5)_{\mathrm{GUT}}$ in order to break the $\mathrm{SU}(5)_{\mathrm{GUT}}$ unified symmetry to $\mathrm{SU}(3)_C\times\mathrm{SU}(2)_L$ possibly with $\mathrm{U}(1)_Y$ as well. Matter multiplets from $\pi_Z^* E$ are characterized as cohomology groups on 4-cycles (7-branes) in F-theory, while those from V, $\wedge^2 V$ and $\wedge^3 V$ are as cohomology on 2-cycles (intersection of 7-branes) in F-theory.

where $\boxplus$ stands for the summation according to the group law of an elliptic curve.

Direct Images and Matter Curves

If an SU(N) vector bundle V is turned on within one of E_8 gauge group of the Heterotic $E_8\times E_8'$ string theory, symmetry group is reduced to $H\subset E_8$ that commutes with the SU(N) in effective theory below the Kaluza–Klein scale. The chiral multiplets in low-energy effective theory are identified with $H^1(Z;\rho(V))$. The correspondence between the representations $\rho(V)$ of V and those of the unbroken symmetry group H is summarized in Table 1.

For a Calabi–Yau 3-fold Z that is an elliptic fibration over a 2-fold B_2, cohomology groups $H^1(Z;\rho(V))$ can be calculated by Leray spectral sequence. One calculates the cohomology in the fiber direction first, $R^i\pi_{Z*}\rho(V)$ $(i=0,1)$, and then the cohomology in the base directions. If $R^0\pi_{Z*}\rho(V)$ vanishes everywhere on B_2, which is often the case, then

$$H^1(Z;\rho(V)) \simeq H^0(B_2; R^1\pi_{Z*}\rho(V)). \tag{15}$$

If one is interested only in the net chirality, i.e. the difference between the number of chiral

multiplets and anti-chiral multiplets in a given representation,

$$\begin{aligned} \chi(\rho(V)) &:= h^1(Z;\rho(V)) - h^1(Z;\rho(V)^\times), \\ &= h^1(Z;\rho(V)) - h^2(Z;\rho(V)), \\ &= -\chi(\rho(V)^\times), \end{aligned} \tag{16}$$

then one has

$$\begin{aligned} \chi(\rho(V)) &= -\chi(Z;\rho(V)), \\ &= -\chi(B_2; R^0\pi_{Z*}\rho(V)) + \chi(B_2; R^1\pi_{Z*}\rho(V)), && (17) \\ &\rightarrow \chi(B_2; R^1\pi_{Z*}\rho(V)) \qquad \text{(if } R^0\pi_{Z*}\rho(V) = 0). && (18) \end{aligned}$$

Suppose that the vector bundle V is given by spectral cover construction from $(C_V, \mathcal{N}_V)$. Let us consider a Fourier–Mukai transform of $\rho(V)$:

$$R^1 p_{1*}\left[p_2^*(\rho(V)) \otimes \mathcal{P}_B^{-1} \otimes \mathcal{O}(-q^* K_{B_2})\right], \tag{19}$$

which is a sheaf on Z, and p_1 and p_2 here are maps in

$$\begin{tikzcd} & Z \times_{B_2} Z \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ Z \arrow[dr, "\pi_Z"'] & & Z \arrow[dl, "\pi_Z"] \\ & B_2 & \end{tikzcd} \tag{20}$$

and $q = \pi_Z \circ p_1 = \pi_Z \circ p_2$. This sheaf is supported only on a codimension-1 subvariety $C_{\rho(V)}$. Unless $C_{\rho(V)}$ contains the zero section σ as an irreducible component, $\rho(V)$ does not contain a trivial bundle when it is restricted on a fiber E_b of a generic point $b \in B_2$. Thus, $R^1\pi_Z{*}\rho(V)$ vanishes on a generic point on B_2; it survives only along a curve

$$\bar{c}_{\rho(V)} = C_{\rho(V)} \cdot \sigma. \tag{21}$$

in B_2 (see also the appendix A). Curves $\bar{c}_{\rho(V)}$ for various representations $\rho(V)$ are called matter curves, because cohomology groups are localized.

The localization of cohomology groups (or matter multiplets that appear in low-energy effective theory) on matter curves is not just an artifact of mathematical calculation. It also has physics meaning. For small elliptic fiber, where F-theory description becomes better,

zero modes of Dirac equation in a given representation $\rho(V)$ have Gaussian profile around a locus where Wilson lines in the elliptic fiber directions vanish, just like the case explained for the T^3-fibration in [9]. Localized massless matter multiplets in Heterotic theory description correspond to those on 7-brane intersection curves in Type IIB / F-theory description.

Suppose that the sheaf (19) on Z is given by a pushforward of a sheaf $\mathcal{N}_{\rho(V)}$ on $C_{\rho(V)}$:

$$R^1 p_{1*}\left[p_2^*(\rho(V)) \otimes \mathcal{P}_B^{-1} \otimes \mathcal{O}(-q^* K_{B_2})\right] = i_{C_{\rho(V)}*}(\mathcal{N}_{\rho(V)}), \tag{22}$$

where $i_{C_{\rho(V)}} : C_{\rho(V)} \hookrightarrow Z$. Then, the direct images $R^1\pi_{Z*}\rho(V)$ are given by pushforwards of sheaves on matter curves [16, 17, 18]:

$$\begin{aligned} R^1\pi_{Z*}(\rho(V)) &= i_{\rho(V)*}\,\mathcal{F}_{\rho(V)}, &(23)\\ \mathcal{F}_{\rho(V)} &= j^*_{\rho(V)}\mathcal{N}_{\rho(V)} \otimes \mathcal{O}(i^*_{\rho(V)}K_{B_2}); &(24)\end{aligned}$$

here, $i_{\rho(V)} : \bar{c}_{\rho(V)} = \sigma \cdot C_{\rho(V)} \hookrightarrow \sigma \simeq B_2$, and $j_{\rho(V)} : \bar{c}_{\rho(V)} = \sigma \cdot C_{\rho(V)} \hookrightarrow C_{\rho(V)}$. Chiral multiplets in low-energy effective theory are characterized as global holomorphic sections of the sheaves $\mathcal{F}_{\rho(V)}$ on the matter curves:

$$H^1(Z;\rho(V)) \simeq H^0(B_2; R^1\pi_{Z*}\rho(V)) \simeq H^0(\bar{c}_{\rho(V)}; \mathcal{F}_{\rho(V)}). \tag{25}$$

The net chirality (17) is now expressed by Euler characteristic on the matter curve:

$$\chi(\rho(V)) = \chi(B_2; R^1\pi_{Z*}\rho(V)) = \chi(\bar{c}_{\rho(V)}; \mathcal{F}_{\rho(V)}). \tag{26}$$

Matter from Bundles in the Fundamental Representation

In the above discussion we have assumed that the sheaf (19) on Z is given by a pushforward of a sheaf on $C_{\rho(V)}$. This is actually a highly non-trivial statement. Even if a sheaf $\mathcal{E}$ on an algebraic variety X is supported on a closed subvariety $i_Y : Y \hookrightarrow X$, it is not true in general that $\mathcal{E}$ is a pushforward of a sheaf $\mathcal{F}$ on Y; $\mathcal{E} = i_{Y*}\mathcal{F}$. It is true that $\mathcal{E} = i_{Y*}\mathcal{F}$ for some $\mathcal{F}$ on Y as a sheaf of Abelian group, but not necessarily as a sheaf of $\mathcal{O}_X$-module. Thus, the discussion after (22) is not necessarily applied immediately for bundles in any representation.

For bundles V in the fundamental representation, however, their Fourier–Mukai transforms are pushforward of the original line bundles $\mathcal{N}_V$ (see section 4 and appendix A). Thus, the discussion all the way down to (26) is applicable. The matter curves $\bar{c}_V = C_V \cdot \sigma$ belong to a topological class

$$\bar{c}_V \in |NK_{B_2} + \eta| \tag{27}$$

because $C_V \in |N\sigma + \pi_Z^*\eta|$, and $\sigma \cdot \sigma = -\sigma \cdot c_1(\mathcal{L}_H) = \sigma \cdot K_{B_2}$ [12].

$R^1\pi_{Z*}V$ is given by a pushforward of a sheaf on $\bar{c}_V$

$$\mathcal{F}_V = j_V^*\mathcal{N} \otimes i_V^*\mathcal{O}(K_{B_2}) = \mathcal{O}\left(i_V^* K_{B_2} + \frac{1}{2} j_V^* r + j_V^*\gamma\right) \tag{28}$$

as a sheaf of $\mathcal{O}_{B_2}$-module. Since the canonical divisor K_{C_V} is also the divisor $C_V|_{\bar{c}_V}$ in a Calabi–Yau 3-fold,

$$\begin{aligned} i_V^* K_{B_2} + \frac{1}{2} j_V^* r &= i_V^* K_{B_2} + \frac{1}{2} j_V^* \left(K_{C_V} - \pi_C^* K_{B_2}\right) = \frac{1}{2}\left(i_V^* K_{B_2} + C_V|_{\bar{c}_V}\right) \\ &= \frac{1}{2}\left(i_V^* K_{B_2} + N_{\bar{c}_V|B_2}\right) = \frac{1}{2} K_{\bar{c}_V}, \end{aligned} \tag{29}$$

where adjunction formula was used for $i_V : \bar{c}_V \hookrightarrow B_2$ [16]. Thus, the sheaf can be rewritten as

$$\mathcal{F}_V = \mathcal{O}\left(\frac{1}{2} K_{\bar{c}_V} + j_V^*\gamma\right), \tag{30}$$

$$\mathcal{F}_{V^\times} = \mathcal{O}\left(\frac{1}{2} K_{\bar{c}_V} - j_V^*\gamma\right); \tag{31}$$

here we determined $\mathcal{F}_{V^\times}$ by replacing γ by $-\gamma$ [17]. It is easy to see that these sheaves satisfy

$$\mathcal{F}_{V^\times} = \mathcal{O}(K_{\bar{c}_V}) \otimes \mathcal{F}_V^{-1}. \tag{32}$$

Massless chiral multiplets from the bundles V and $V^\times$ are now given by independent global holomorphic sections of $\mathcal{F}_V$ and $\mathcal{F}_{V^\times}$, respectively. If one is interested only in the difference between the numbers of those chiral multiplets, the net chirality is obtained by Riemann–Roch theorem [16, 17]:

$$\begin{aligned} \chi(V) &= \chi(\bar{c}_V; \mathcal{F}_V), && (33) \\ &= [1 - g(\bar{c}_V)] + \deg\left(K_{B_2} + \frac{1}{2} j_V^* r\right) + \int_{\bar{c}_V} j_V^*\gamma, && (34) \\ &= [1 - g(\bar{c}_V)] + \frac{1}{2}\deg K_{\bar{c}_V} + \int_{\bar{c}_V} j_V^*\gamma, && (35) \\ &= \int_{\bar{c}_V} j_V^*\gamma = \bar{c}_V \cdot \gamma. && (36) \end{aligned}$$

It is reasonable that the final result is proportional to γ, because we know that $\chi(V) = -\chi(V^\times)$, and $V \leftrightarrow V^\times$ corresponds to $\gamma \leftrightarrow -\gamma$ and $\mathcal{P}_B \leftrightarrow \mathcal{P}_B^{-1}$ [17].

3 Bundles Trivial in the Fiber Direction

In this section we briefly discuss the cohomology groups $H^i(Z;\pi_Z^*E)$, where Z is an elliptic fibration $\pi_Z : Z \rightarrow B_2$, and we consider a bundle given by a pullback of a bundle E on B_2. Bundles given by π_Z^* are trivial in the fiber direction, and hence $R^0\pi_{Z*}(\pi_Z^*E)$ on B_2 does not vanish, and $R^1\pi_{Z*}(\pi_Z^*E)$ is not supported on a curve in B_2, either. Thus, the cohomology groups of the bundles π_Z^*E are not described in the same way as those of such bundles as V, $\wedge^2 V$ and $\wedge^3 V$. We need to express $H^i(Z;\pi_Z^*E)$ $(i = 1, 2)$ in terms of cohomology groups of $R^p\pi_{Z*}(\pi_Z^*E)$ $(p = 0, 1)$, so that those expressions are interpreted in F-theory.

This issue has been discussed in the footnote 13 of [14]. (See also [10].) Here, we add a minor comment to the description given there.

First, note that

$$
\begin{aligned}
R^0\pi_{Z*}\left(\pi_Z^*E\right) &\simeq E, && (37)\\
R^1\pi_{Z*}\left(\pi_Z^*E\right) &\simeq E\otimes\mathcal{L}_H^{-1} \simeq E\otimes\mathcal{O}(K_{B_2}), && (38)
\end{aligned}
$$

where the Calabi–Yau condition of $\pi_Z : Z \rightarrow B_2$ is used in the last equality. Thus,

$$
\begin{aligned}
H^0(Z;\pi_Z^*E) &\simeq H^0(B_2;E), && (39)\\
\left[H^0(Z;\pi_Z^*E^\times)\right]^\times \simeq H^3(Z;\pi_Z^*E) &\simeq H^2(B_2;E\otimes\mathcal{O}(K_{B_2})) \simeq [H^0(B_2;E^\times)]^\times. && (40)
\end{aligned}
$$

Since these cohomology groups correspond to massless gauginos at low energy, one can assume that those groups are trivial when one is concerned with matter multiplets. Using the spectral sequence, one can see that the two other cohomology groups $H^r(Z;\pi_Z^*E)$ $(r = 1, 2)$ satisfy

$$
\begin{aligned}
&0 \rightarrow H^1(B_2;E) \rightarrow H^1(Z;\pi_Z^*E) \rightarrow H^0(B_2;E\otimes\mathcal{O}(K_{B_2})) \rightarrow H^2(B_2;E), && (41)\\
&H^0(B_2;E\otimes\mathcal{O}(K_{B_2})) \rightarrow H^2(B_2;E) \rightarrow H^2(Z;\pi_Z^*E) \rightarrow H^1(B_2;E\otimes\mathcal{O}(K_{B_2})) \rightarrow 0. && (42)
\end{aligned}
$$

In the spectral sequence calculation of cohomology groups, $E_2^{p,q} = H^p(B_2;R^q\pi_{Z*}\pi_Z^*E)$, and $d_2 : E_2^{p,q} \rightarrow E_2^{p+2,q-1}$ for $(p,q) = (0,1)$ determines the map

$$
d_2 : H^0(B_2;E\otimes\mathcal{O}(K_{B_2})) \rightarrow H^2(B_2;E) \simeq \left[H^0(B_2;E^\times\otimes\mathcal{O}(K_{B_2}))\right]^\times \tag{43}
$$

used in (41, 42).

It thus follows that

$$
\begin{aligned}
h^1(Z;\pi_Z^*E) &= h^1(B_2;E) + \ker\, d_2, && (44)\\
h^2(Z;\pi_Z^*E) &= h^1(B_2;E\otimes\mathcal{O}(K_{B_2})) + \mathrm{coker}\, d_2, && (45)\\
&= h^1(B_2;E^\times) + \mathrm{coker}\, d_2, && (46)
\end{aligned}
$$

where d_2 is the one in (43). If d_2 is trivial (including cases where either $h^0(B_2; E \otimes \mathcal{O}(K_{B_2})) = 0$ or $h^0(B_2; E^\times \otimes \mathcal{O}(K_{B_2})) = 0$), the results in [14] follow:

$$
\begin{aligned}
h^1(Z; \pi_Z^* E) &= h^1(B_2; E) + h^0(B_2; E \otimes \mathcal{O}(K_{B_2})), &(47)\\
h^1(Z; \pi_Z^* E^\times) = h^2(Z; \pi_Z^* E) &= h^1(B_2; E \otimes \mathcal{O}(K_{B_2})) + h^2(B_2; E), &(48)\\
&= h^1(B_2; E^\times) + h^0(B_2; E^\times \otimes \mathcal{O}(K_{B_2})). &(49)
\end{aligned}
$$

For a general d_2, (44, 46) are the right expressions for the number of massless matter multiplets from $\pi_Z^* E$. This means that some degrees of freedom in $H^0(B_2; E \otimes \mathcal{O}(K_{B_2}))$ and $H^0(B_2; E^\times \otimes \mathcal{O}(K_{B_2}))$ are paired up and do not remain in the low-energy spectrum. One might phrase this phenomenon as those degrees of freedom having "masses." It should be noted that all the degrees of freedom in $H^1(B_2; E)$ and $H^1(B_2; E^\times)$ do not have such "masses." We do not study the detail of the map d_2 based on explicit examples. Such "masses" may be understood as a kind of obstruction in geometry. We leave these interesting questions as open problems for the future.

The structure group of a bundle E can be chosen so that the unbroken symmetry H in Table 1 is reduced to whatever one likes, say $\mathrm{SU}(5)_{\mathrm{GUT}}$ or $\mathrm{SU}(3) \times \mathrm{SU}(2)$. The irreducible decomposition of $\mathbf{adj}.H$ under the structure group of E and the true unbroken symmetry may contain a pair of vector-like representations, $(\rho(E), repr.)$–$(\rho(E)^\times, repr.^\times)$. For such a pair, the net chirality is calculated by

$$
\begin{aligned}
\chi(\rho(E)) &:= h^1(Z; \pi_Z^* \rho(E)) - h^1(Z; \pi_Z^* \rho(E)^\times), &(50)\\
&= -\chi(Z; \pi_Z^* \rho(E)), &(51)\\
&= -\chi(B_2; \rho(E)) + \chi(B_2; \rho(E) \otimes \mathcal{O}(K_{B_2})), &(52)\\
&= -\int_{B_2} c_1(TB_2) \wedge c_1(\rho(E)). &(53)
\end{aligned}
$$

Rank of the map d_2 in (43) does not matter here.

The chirality formula (53) can also be obtained from the discussion reviewed in the previous section [1]. The bundle $\pi_Z^* \rho(E)$ is regarded as a Fourier–Mukai transform of $(C_{\rho(E)}, \mathcal{N}_{\rho(E)}) = (\sigma, \rho(E))$. Thus, the matter curve is formally given by $C_{\rho(E)} \cdot \sigma$ which belongs to a class of K_{B_2}. Since the ramification divisor of $\pi_C : C \to B_2$ is trivial, one finds (i) from the argument in (29) that K_{B_2} is half the canonical divisor of the "matter curve" $\bar{c}_{\rho(E)} \sim K_{B_2}$ in B_2, and (ii) that $\mathcal{N}_{\rho(E)} \otimes \mathcal{O}(r/2)^{-1} = \rho(E)$. Therefore,

$$
\chi(\rho(E)) = \int_{K_{B_2}} c_1(E) = -\int_{B_2} c_1(TB_2) \wedge c_1(\rho(E)), \tag{54}
$$

reproducing (53).

4 Analysis of $R^1\pi_{Z*}\wedge^2 V$

Not all the chiral multiplets in low-energy effective theory are identified with cohomology groups of bundle V (or $V^\times$) in the fundamental (anti-fundamental) representation, as we see in Table 1. In order to obtain description of all kinds of matter multiplets in F-theory, we also need to determine the sheaves $R^1\pi_{Z*}\rho(V)$ for bundles associated with $\rho(V)=\wedge^2 V$ and $\wedge^3 V$. As we have emphasized in Introduction, the Higgs multiplets and $\bar{\mathbf{5}}=(\bar{D},L)$ in the $\mathrm{SU}(5)_{\mathrm{GUT}}$-$\mathbf{5}+\bar{\mathbf{5}}$ representations originate from the bundle $\wedge^2 V$, and the Higgs multiplet in the SO(10)-$\mathbf{10}=\mathbf{vec}$. representation from $\wedge^2 V$. Thus, it is important to determine $R^1\pi_{Z*}\wedge^2 V$ in order to understand Yukawa couplings of quarks and leptons in F-theory language.

For the bundles V (or $V^\times$), the generic element of a topological class of spectral surface $|N\sigma+\pi_Z^*\eta|$ is smooth,[4] and the transverse coordinate of C_V in Z can be chosen at any points on C_V. This property can be used to show that the Fourier–Mukai transform of V is given by a pushforward of a sheaf on C_V as a sheaf of $\mathcal{O}_Z$ module (see the appendix A). Furthermore, the rank of fiber of the Fourier–Mukai transform never jumps on C_V and the sheaf is the locally-free rank-1 sheaf $\mathcal{N}_V$ itself.

For the bundles $\wedge^2 V$ (or $\wedge^2 V^\times$), on the other hand, $C_{\wedge^2 V}$ is not necessarily smooth, even if C_V is. Here, we denote by $C_{\wedge^2 V}$ the support of Fourier–Mukai transform (19) of $\rho(V)=\wedge^2 V$. Suppose that $C_V|_{E_b}$ for a point $b\in B_2$ consists of N points $\{p_i\}_{i=1,\cdots N}$. Then, $C_{\wedge^2 V}|_{E_b}$ is given by $\{p_i\boxplus p_j\}_{1\leq i<j\leq N}$. At a generic point $b\in B_2$, the $N(N-1)/2$ points $p_i\boxplus p_j$ $(i<j)$ in elliptic fiber E_b are all different, and $C_{\wedge^2 V}$ is a smooth degree $N(N-1)/2$ cover. For these points, the arguments of the appendix A can be used to show that there a locally free rank-1 sheaf $\mathcal{N}_{\wedge^2 V}$ exists on $C_{\wedge^2 V}$ (locally around smooth points in $C_{\wedge^2 V}$), and the Fourier–Mukai transform of $\wedge^2 V$ is represented as the pushforward of $\mathcal{N}_{\wedge^2 V}$ as a sheaf of $\mathcal{O}_Z$-module. But, on a codimension-1 locus of $C_{\wedge^2 V}$, $C_{\wedge^2 V}$ may become singular [10], and a little more attention must be paid.

We will describe a rough sketch of how to determine $R^1\pi_{Z*}\wedge^2 V$ in this section, beginning with how to deal with such singularities. Details of $R^1\pi_{Z*}\wedge^2 V$ are deferred to the next section. Since some crucial aspects of $R^1\pi_{Z*}\wedge^2 V$ depend very much on the rank of V, we will provide detailed description of $R^1\pi_{Z*}\wedge^2 V$ for the rank of V between 2 and 6 in the next section. Once we see how to deal with $R^1\pi_{Z*}\wedge^2 V$, it is rather straightforward to find how

[4] Some conditions have to be imposed on the divisor η. See [22].

to deal with $R^1\pi_{Z*} \wedge^3 V$. We will only discuss $R^1\pi_{Z*} \wedge^3 V$ in section 5.5.

4.1 Resolving Double-Curve Singularity of $C_{\wedge^2 V}$

$C_{\wedge^2 V}$ is described locally as $N(N-1)/2$ surfaces that $p_i \boxplus p_j$ $(i < j)$ scan. $C_{\wedge^2 V}$ has a double-curve singularity if $p_i \boxplus p_j$ $(i < j)$ and $p_k \boxplus p_l$ $(k < l, \{i,j\} \cap \{k,l\} = \phi)$ become equal. It is not obvious how to choose a coordinate in Z that is normal to $C_{\wedge^2 V}$ along the double-curve locus, and the argument in the appendix is not readily applicable.

In a local neighborhood of the double curve, $C_{\wedge^2 V}$ consists of two irreducible components, $C_{(ij)}$ and $C_{(kl)}$, and their intersection is the double-curve singularity. $C_{(ij)}$ and $C_{(kl)}$ are surfaces scanned in Z by $p_i \boxplus p_j$ and $p_k \boxplus p_l$. $\rho(V) = \wedge^2 V$ can be regarded locally as direct sum of $\mathcal{O}(C_{(ij)} - \sigma)$, $\mathcal{O}(C_{(kl)} - \sigma)$ and others. Its Fourier–Mukai transform in (19) is given by a sum of the above two summands. The Fourier–Mukai transform of the two summands $\mathcal{O}(C_{(ij)} - \sigma)$ and $\mathcal{O}(C_{(kl)} - \sigma)$ is expressed locally as

$$R^1 p_{1*}\left[\mathcal{O}(C_{(ij)} - \sigma) \otimes \mathcal{P}_B^{-1} \otimes \mathcal{O}(-q^* K_{B_2})\right] = i_{C_{(ij)}*}\mathcal{O}_{C_{(ij)}}, \tag{55}$$
$$R^1 p_{1*}\left[\mathcal{O}(C_{(kl)} - \sigma) \otimes \mathcal{P}_B^{-1} \otimes \mathcal{O}(-q^* K_{B_2})\right] = i_{C_{(kl)}*}\mathcal{O}_{C_{(kl)}}. \tag{56}$$

Here, $i_{C_{\wedge^2 V}} : C_{\wedge^2 V} \hookrightarrow Z$ (which is different from previously defined $i_{\wedge^2 V} : \bar{c}_{\wedge^2 V} \hookrightarrow \sigma$), and

$$\nu_{C_{ij}} : C_{(ij)} \hookrightarrow C_{\wedge^2 V}, \qquad i_{C_{(ij)}} = i_{C_{\wedge^2 V}} \circ \nu_{C_{ij}}, \tag{57}$$
$$\nu_{C_{kl}} : C_{(kl)} \hookrightarrow C_{\wedge^2 V}, \qquad i_{C_{(kl)}} = i_{C_{\wedge^2 V}} \circ \nu_{C_{kl}}. \tag{58}$$

Therefore, the Fourier–Mukai transform of $\wedge^2 V$ is

$$R^1 p_{1*}\left[p_2^*(\wedge^2 V) \otimes \mathcal{P}_B^{-1} \otimes \mathcal{O}(-q^* K_{B_2})\right] \simeq i_{C_{\wedge^2 V}*}\left(\nu_{C_{ij}*}\mathcal{O}_{C_{(ij)}} \oplus \nu_{C_{kl}*}\mathcal{O}_{C_{(kl)}}\right) \tag{59}$$

locally along a double-curve singularity. Thus, it is given by a pushforward of a sheaf $\mathcal{N}_{\wedge^2 V}$ on $C_{\wedge^2 V}$ as a sheaf of $\mathcal{O}_Z$-module. The sheaf $\mathcal{N}_{\wedge^2 V}$ is the object inside the parenthesis on the right hand side.

The sheaf $\mathcal{N}_{\wedge^2 V}$ is not locally free along the double-curve singularity. The rank of fiber jumps up there. But we already know that the sheaf $\mathcal{N}_{\wedge^2 V}$ is given by a pushforward of locally-free rank-1 sheaf via

$$\nu_{C_{\wedge^2 V}} : \widetilde{C}_{\wedge^2 V} = C_{(ij)} \coprod C_{(kl)} \to C_{(ij)} \cup C_{(kl)} = C_{\wedge^2 V}. \tag{60}$$

The map $\nu_{C_{\wedge^2 V}}$ is determined by $\nu_{C_{ij}} \coprod \nu_{C_{kl}}$. Note that $\widetilde{C}_{\wedge^2 V} := C_{(ij)} \coprod C_{(kl)}$ is the resolution of double-curve singularity in $C_{\wedge^2 V}$. Therefore, the discussion so far means that there exists

a locally free rank-1 sheaf $\widetilde{\mathcal{N}}_{\wedge^2 V}$ on the resolved $\widetilde{C}_{\wedge^2 V}$ such that

$$\mathcal{N}_{\wedge^2 V} = \nu_{C_{\wedge^2 V} *} \widetilde{\mathcal{N}}_{\wedge^2 V}. \tag{61}$$

We have seen that the sheaf $\mathcal{N}_{\wedge^2 V}$ exists on $C_{\wedge^2 V}$ and (22) is satisfied as a sheaf of $\mathcal{O}_Z$ module. Thus, the discussion around equations (22–26) is applied for the bundles $\rho(V) = \wedge^2 V$ and $\wedge^2 V^\times$ as well. In particular, the sheaf on the matter curve $\bar{c}_{\wedge^2 V}$ is given by

$$\mathcal{F}_{\wedge^2 V} = j^*_{\wedge^2 V} \mathcal{N}_{\wedge^2 V} \otimes i^*_{\wedge^2 V} \mathcal{O}(K_{B_2}). \tag{62}$$

We introduce the notion of covering matter curve, which turns out to be very important in characterizing matter multiplets in F-theory. The covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$ is defined as the set-theoretic inverse image of the matter curve $\bar{c}_{\wedge^2 V}$ in $\widetilde{C}_{\wedge^2 V}$. That is, $\tilde{\bar{c}}_{\wedge^2 V} := \nu^{-1}_{C_{\wedge^2 V}}(\bar{c}_{\wedge^2 V})$. Since the matter curve $\bar{c}_{\wedge^2 V}$ is also regarded as a divisor $\sigma|_{C_{\wedge^2 V}}$ in $C_{\wedge^2 V}$, the covering matter curve is also regarded as a divisor $\nu^*_{C_{\wedge^2 V}}(\sigma)$ on $\widetilde{C}_{\wedge^2 V}$. Using a locally rank-1 sheaf $\widetilde{\mathcal{N}}_{\wedge^2 V}$ on $\widetilde{C}_{\wedge^2 V}$, a locally free rank-1 sheaf $\widetilde{\mathcal{F}}_{\wedge^2 V}$ can be defined on the covering matter curve:

$$\widetilde{\mathcal{F}}_{\wedge^2 V} = \tilde{j}^*_{\wedge^2 V} \widetilde{\mathcal{N}}_{\wedge^2 V} \otimes \tilde{i}^*_{\wedge^2 V} \mathcal{O}(K_{B_2}), \tag{63}$$

where $\tilde{j}_{\wedge^2 V} : \tilde{\bar{c}}_{\wedge^2 V} \hookrightarrow \widetilde{C}_{\wedge^2 V}$, $\nu_{\bar{c}_{\wedge^2 V}} := \nu_{C_{\wedge^2 V}}|_{\tilde{\bar{c}}_{\wedge^2 V}}$, and $\tilde{i}_{\wedge^2 V} := i_{\wedge^2 V} \circ \nu_{\bar{c}_{\wedge^2 V}} : \tilde{\bar{c}}_{\wedge^2 V} \hookrightarrow \sigma \simeq B_2$. The sheaf $\mathcal{F}_{\wedge^2 V}$ on the matter curve $\bar{c}_{\wedge^2 V}$ is given by

$$\mathcal{F}_{\wedge^2 V} = \nu_{\bar{c}_{\wedge^2 V} *} \widetilde{\mathcal{F}}_{\wedge^2 V}. \tag{64}$$

Although we have dealt with double-curve singularities on $C_{\wedge^2 V}$, there can still be other types of singularities on $C_{\wedge^2 V}$. For example, there may be codimension-2 singularities on $C_{\wedge^2 V}$. Thus, the argument in section 4.1 is not regarded as a complete proof of the existence of $\mathcal{N}_{\wedge^2 V}$ or the existence of $\widetilde{\mathcal{N}}_{\wedge^2 V}$ and its locally-free rank-1 nature. For practical purposes, however, we only need to know $R^1 \pi_{Z*} \wedge^2 V$ along the matter curves. Codimension-1 singularities such as double curve on $C_{\wedge^2 V}$ may be encountered somewhere along the matter curve $\bar{c}_{\wedge^2 V}$ [10], but codimension-2 singularities of $C_{\wedge^2 V}$ are seldom exactly on the matter curve. Thus, an analysis of codimension-2 singularities of $C_{\wedge^2 V}$ is not required for the generic case. We will see, however, that codimension-2 singularities inevitably show up on the matter curve $\bar{c}_{\wedge^2 V}$ when rank $V = 4, 6$. We will deal with such exceptional cases separately in sections 5.3 and 5.5.

4.2 Determining $\widetilde{\mathcal{N}}_{\wedge^2 V}$ in Terms of $\mathcal{N}_V$

Even after we find that a sheaf $\mathcal{N}_{\wedge^2 V}$ exists and (22) is satisfied as a sheaf of $\mathcal{O}_Z$ module, we still face a theoretical challenge. How is $\mathcal{N}_{\wedge^2 V}$ (or $\widetilde{\mathcal{N}}_{\wedge^2 V}$) expressed in terms of the original spectral data $(C_V, \mathcal{N}_V)$? Pioneering work was done in [18, 19]. Our presentation in the following is basically along their idea,[5] but we introduce a little modification for a couple of reasons. First, we will obtain sheaves $\tilde{\mathcal{N}}_{\wedge^2 V}$ and $\widetilde{\mathcal{F}}_{\wedge^2 V}$ on the covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$, instead of $\mathcal{F}_{\wedge^2 V}$ on the matter curve $\bar{c}_{\wedge^2 V}$. By doing so, much clearer description of the direct image $R^1\pi_{Z*} \wedge^2 V$ is obtained. The other reason for modification is that we are not assuming that $\mathcal{N}_V|_D$ is invariant under τ that flips the sign of the coordinate y. D is a curve on C_V; we will explain it later.

Since (22) for $\rho(V) = \wedge^2 V$ is the definition of $\mathcal{N}_{\wedge^2 V}$, it follows that

$$\mathcal{N}_{\wedge^2 V} = i^*_{C_{\wedge^2 V}} R^1 p_{1*} \left[p_2^*(\wedge^2 V) \otimes \mathcal{P}_{B_2}^{-1} \otimes \mathcal{O}(-q^* K_{B_2}) \right]. \tag{65}$$

What we really need is its restriction on $\bar{c}_{\wedge^2 V}$, and hence

$$\begin{aligned}
\mathcal{F}_{\wedge^2 V} &= \mathcal{N}_{\wedge^2 V}|_{\bar{c}_{\wedge^2 V}} \otimes i^*_{\wedge^2 V}\mathcal{O}(K_{B_2}), \\
&= (i_{C_{\wedge^2 V}} \circ j_{\wedge^2 V})^* R^1 p_{1*} \left[p_2^*(\wedge^2 V) \otimes \mathcal{P}_{B_2}^{-1} \otimes \mathcal{O}(-q^* K_{B_2}) \right] \otimes i^*_{\wedge^2 V}\mathcal{O}(K_{B_2}), \\
&= (i_{C_{\wedge^2 V}} \circ j_{\wedge^2 V})^* R^1 p_{1*} \left[p_2^*(\wedge^2 V) \right], \\
&= R^1 p_{1Y*} \left[\wedge^2 V|_Y \right];
\end{aligned} \tag{66}$$

here, $Y := \bar{c}_{\wedge^2 V} \times_{B_2} Z = \pi_Z^{-1}(\bar{c}_{\wedge^2 V})$. In the third equality, we used the property that $\mathcal{P}_{B_2}$ is trivial when it is restricted to a zero section [12], and in the last equality the base change theorem associated with the commutative diagram

$$\begin{tikzcd}
Y := \bar{c}_{\wedge^2 V} \times_{B_2} Z \arrow[rr, hook] \arrow[d, "p_{1Y}"'] & & Z \times_{B_2} Z \arrow[d, "p_1"] \\
\bar{c}_{\wedge^2 V} \arrow[r, hook, "j_{\wedge^2 V}"'] & C_{\wedge^2 V} \arrow[r, hook, "i_{C_{\wedge^2 V}}"'] & Z
\end{tikzcd} . \tag{67}$$

This is the standard procedure used in [16, 17, 18].

[5] One of the authors (TW) thanks Ron Donagi for explaining the idea of [18] (March, 2006).

The rank-N bundle $V|_Y$ is given by a Fourier–Mukai transform of $\mathcal{N}_V|_{C_V\cdot Y}$:

$$
\begin{tikzcd}
 & (C_V\cdot Y)\times_{\bar{c}_{\wedge^2 V}} Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
C_V\cdot Y \arrow[dr, "\pi_C|_{C_V\cdot Y}"'] & & Y \arrow[dl, "\pi_Y"] \\
 & \bar{c}_{\wedge^2 V} &
\end{tikzcd}, \qquad V|_Y = p_{2*}\left(\mathcal{P}_{B_2}\otimes p_1^*(\mathcal{N}_V|_{C_V\cdot Y})\right). \tag{68}
$$

The spectral curve $C_V\cdot Y$ is a degree-N cover over $\bar{c}_{\wedge^2 V}$.

Let $C_V|_{E_b}$ be a collection of N points $\{p_i\}_{i=1,\cdots,N}$. For a point $b\in\bar{c}_{\wedge^2 V}\subset C_{\wedge^2 V}$, some pairs of the N points, e.g., p_k and p_l, satisfy $p_k\boxplus p_l=e_0$. Such points in $C_V\cdot Y$ form a curve D, and others form a curve D'.

$$
C_V\cdot Y = D + D'. \tag{69}
$$

By the definition of D, the following diagram commutes [19],[6]

$$
\begin{tikzcd}
D \arrow[r, "\tilde{\pi}_D"] \arrow[dr, "\pi_D"'] & \tilde{\bar{c}}_{\wedge^2 V} \arrow[d, "\nu_{\bar{c}_{\wedge^2 V}}"] \\
 & \bar{c}_{\wedge^2 V}
\end{tikzcd} \tag{70}
$$

and $\tilde{\pi}_D$ is a degree-2 cover, and π_D is a restriction of π_C on D. If $b\in\bar{c}_{\wedge^2 V}\hookrightarrow\sigma$ is on the double-curve singularity of $C_{\wedge^2 V}$, then there are four points $p_{i,j,k,l}$, satisfying $p_i\boxplus p_j=e_0$ and $p_k\boxplus p_l=e_0$. In the covering matter curve, the inverse image of b, that is, $\nu^{-1}_{\bar{c}_{\wedge^2 V}}(b)$, consists of two points. Two points $p_{i,j}\in D$ are mapped by $\tilde{\pi}_D$ to one of the two points in $\nu^{-1}_{\bar{c}_{\wedge^2 V}}(b)$, and $p_{k,l}\in D$ to the other. Although all the four points are mapped to $b\in\bar{c}_{\wedge^2 V}$ by π_D, $\tilde{\pi}_D$ remains strictly a degree-2 cover everywhere on $\tilde{\bar{c}}_{\wedge^2 V}$.

The Fourier–Mukai transform of $\mathcal{N}_V|_D$ on a degree-2 cover spectral curve $\tilde{\pi}_D: D\to\tilde{\bar{c}}_{\wedge^2 V}$ gives a rank-2 bundle W_2:

$$
\begin{tikzcd}
 & D\times_{\tilde{\bar{c}}_{\wedge^2 V}}\tilde{Y} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
D \arrow[dr, "\tilde{\pi}_D"'] & & \tilde{Y} \arrow[dl, "p_{1\tilde{Y}}"] \\
 & \tilde{\bar{c}}_{\wedge^2 V} &
\end{tikzcd}, \qquad W_2 = p_{2*}\left(\mathcal{P}_B|_{D\times_{\tilde{\bar{c}}_{\wedge^2 V}}Y}\otimes p_1^*(\mathcal{N}_V|_D)\right), \tag{71}
$$

[6] In [19], D corresponds to our $C_V\cdot Y$, and D' to our D. The covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$ introduced in this article is essentially the same as D'/τ in [19]. See footnote 19 for more about the relation between D'/τ in [19] and $\tilde{\bar{c}}_{\wedge^2 V}$ here.

where $\tilde{Y} := \tilde{\bar{c}}_{\wedge^2 V} \times_{\bar{c}_{\wedge^2 V}} Y$. The pushforward of this rank-2 bundle W_2 through projection $\nu_Y : \tilde{Y} = \tilde{\bar{c}}_{\wedge^2 V} \times_{\bar{c}_{\wedge^2 V}} Y \to Y$ defines a subsheaf of $V|_Y$.

For a point $b \in \bar{c}_{\wedge^2 V} \subset C_{\wedge^2 V}$ that is not on the double-curve locus, $H^1(E_b; \wedge^2 V|_{E_b})$ comes from $H^1(E_b; \wedge^2(\nu_{Y*} W_2)|_{E_b}) = H^1(E_b; \wedge^2 W_2|_{E_b})$. For a point $b \in \bar{c}_{\wedge^2 V}$ on the double-curve singularity of $C_{\wedge^2 V}$, however, there are two independent contributions corresponding to $H^1(E_b; \wedge^2 W_2|_{E_{\tilde{b}}})$ for two points $\tilde{b} \in \nu^{-1}_{\bar{c}_{\wedge^2 V}}(b)$. We introduced the covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$ in order to resolve these two contributions. The locally free rank-1 sheaf $\widetilde{\mathcal{N}}_{\wedge^2 V}|_{\tilde{\bar{c}}_{\wedge^2 V}}$ (and $\widetilde{\mathcal{F}}_{\wedge^2 V}$, consequently) is obtained by assigning them to the corresponding two points $\tilde{b}$ on $\tilde{\bar{c}}_{\wedge^2 V}$. Therefore,

$$\begin{aligned} \widetilde{\mathcal{F}}_{\wedge^2 V} &= \widetilde{\mathcal{N}}_{\wedge^2 V}|_{\tilde{\bar{c}}_{\wedge^2 V}} \otimes \tilde{\imath}^*_{\wedge^2 V} \mathcal{O}(K_{B_2}), \\ &= R^1 p_{1\tilde{Y}*} \left[\wedge^2 W_2 \right]. \end{aligned} \tag{72}$$

The line bundle $\wedge^2 W_2$ is trivial in the fiber direction of $p_{1\tilde{Y}}$. Thus, it is regarded as a Fourier–Mukai transform of $(C_{\wedge^2 W_2}, \mathcal{N}_{\wedge^2 W_2}) = (\sigma, \widetilde{\mathcal{N}}_{\wedge^2 V}|_{\tilde{\bar{c}}_{\wedge^2 V}})$. It then follows that

$$\wedge^2 W_2 = p^*_{1\tilde{Y}}(\mathcal{N}_{\wedge^2 W_2}). \tag{73}$$

Thus,

$$\widetilde{\mathcal{F}}_{\wedge^2 V} = \mathcal{N}_{\wedge^2 W_2} \otimes \mathcal{L}_H^{-1} = \mathcal{N}_{\wedge^2 W_2} \otimes \tilde{\imath}^*_{\wedge^2 V} \mathcal{O}(K_{B_2}). \tag{74}$$

Now it is useful to remember that the first Chern class of the line bundle $\wedge^2 W_2$ is

$$\begin{aligned} c_1(\wedge^2 W_2) = c_1(W_2) &= p^*_{1\tilde{Y}} \tilde{\pi}_{D*} \left(c_1(\mathcal{N}_V|_D) - \frac{1}{2} R \right), &(75) \\ &= p^*_{1\tilde{Y}} \tilde{\pi}_{D*} \left(\gamma|_D + \frac{1}{2}(r|_D - R) \right), &(76) \end{aligned}$$

just like in (8). Here, $R := K_D - \tilde{\pi}^*_D K_{\tilde{\bar{c}}_{\wedge^2 V}}$ is the ramification divisor on D associated with the projection $\tilde{\pi}_D : D \to \tilde{\bar{c}}_{\wedge^2 V}$. Thus,[7] by dropping $p^*_{1\tilde{Y}}$ from (73) and (76),

$$\widetilde{\mathcal{F}}_{\wedge^2 V} = \mathcal{N}_{\wedge^2 W_2} \otimes \tilde{\imath}^*_{\wedge^2 V} \mathcal{O}(K_{B_2}) = \mathcal{O}\left(\tilde{\imath}^*_{\wedge^2 V} K_{B_2} + \tilde{\pi}_{D*} \left(\gamma|_D + \frac{1}{2}(r|_D - R) \right) \right). \tag{77}$$

[7]Equations (3.56) and (3.57) in [19] would be consistent with (77), if the sign of the $R/2$ terms in the equations were opposite.

5 In-Depth Analysis of Associated Bundles

In this section, we will study direct images $R^1\pi_{Z*}\wedge^2 V$ for rank $V = 2, 3, 4, 5, 6$, and $R^1\pi_{Z*}\wedge^3 V$ for rank $V = 6$. Apart from $\wedge^2 V$ for a rank-2 bundle V, all those associated bundles are non-trivial in the fiber direction, and those direct images have their supports on matter curves. The description of sheaves on the matter curves obtained in this section in Heterotic description are translated into F-theory language in later sections.

Before we commit ourselves to individual cases, we quote some results from [18] that are useful in this section regardless of the rank of V. First, the spectral surface $C_{\wedge^2 V}$ belongs to a class

$$C_{\wedge^2 V} \in \left| \frac{N(N-1)}{2}\sigma + (N-2)\pi_Z^*\eta \right|. \tag{78}$$

The coefficient $N(N-1)/2$ of the first term is the rank of $\wedge^2 V$, and $(N-2)$ for the second term twice the Dynkin index of the rank-2 anti-symmetric tensor representation of SU(N). When the matter curve is given by $C_{\wedge^2 V} \cdot \sigma$, (there is an exception; see sections 5.3)

$$C_{\wedge^2 V} \cdot \sigma \in \left| \frac{N(N-1)}{2} K_B + (N-2)\eta \right| \subset B_2. \tag{79}$$

A curve D in section 4 belongs to a topological class

$$D \in \left|\sigma \cdot [N(N-1)K_{B_2} + 2(N-2)\eta] + \pi_Z^* [\eta \cdot (3K_{B_2} + \eta)] \right|. \tag{80}$$

See [18] for why this is the case.

5.1 Rank-2 Vector Bundles

The $\wedge^2 V$ bundle for a rank-2 bundle V is exceptional in this section, because $\wedge^2 V = \det V$ is trivial in the fiber direction for V given by spectral cover construction. If the structure group of V is SU(2), we have nothing non-trivial to say because $\wedge^2 V = \mathcal{O}_Z$. If the structure group is U(2), then $\wedge^2 V = \pi_Z^* E$ for some line bundle E on the base manifold B_2. Thus, this case is a special case of what we discussed in section 3.

U(2) bundles have appeared in phenomenological applications. In [1], for example, a rank-5 bundle $U_3 \oplus U_2$ was considered for SU(5)$_{\rm GUT}$ unified theories, where U_3 is a rank-3 bundle with the structure group U(3), and U_2 a rank-2 bundle with U(2). By considering a vector bundle in such a semi-stable limit, and some controlled deformation from this limit, one can bring dimension-4 and dimension-5 proton decay problems under control [1, 23, 24].

The up-type Higgs multiplet was identified with $H^2(Z; \wedge^2 U_2^\times)$ and its F-theory dual. Thus, it is not without phenomenological motivation to provide an F-theory description of $\wedge^2 V$ of a $\mathrm{U}(2)$ bundle V.

Since $\wedge^2 V$ is trivial in the fiber direction, anything written in section 3 apply here, with

$$E = \mathcal{O}(\pi_{C*}\gamma_2), \tag{81}$$

where γ_2 is γ in (9) for the bundle U_2. This line bundle E has a structure group U(1) in the commutant of SU(2) in E_8, which is E_7. In the case an SU(5) bundle $U_3 \oplus U_2$ is considered in Heterotic theory compactification, for example, then the line bundle

$$\wedge^2 U_2 = \det U_2 = (\det U_3)^{-1} = \pi_Z^* E \tag{82}$$

has its U(1) structure group in the commutant of $\mathrm{SU}(3) \times \mathrm{SU}(2)$ in E_8, which is now $\mathrm{SU}(6) \supset \mathrm{SU}(5)_{\mathrm{GUT}}$. The structure group is now the $\mathrm{U}(1)$ direction in SU(6) that also commutes with $\mathrm{SU}(5)_{\mathrm{GUT}}$.

5.2 Rank-3 Vector Bundles

Next, let us consider a rank-3 vector bundle V with the data $(C_V, \mathcal{N}_V)$. Note that $\wedge^2 V \simeq V^\times \otimes (\det\ V)$. Since $\det V$ line bundle is trivial in the fiber direction, the spectral surface of $\wedge^2 V$ is the same as that of $V^\times$. If the spectral surface of V is given by a zero locus of

$$s = a_0 + a_2 x + a_3 y, \tag{83}$$

then the spectral surface $C_{V^\times} = C_{\wedge^2 V}$ is given by the zero locus of

$$s^\times = a_0 + a_2 x - a_3 y, \tag{84}$$

flipping the sign of terms containing one y from (83). Thus, $C_{\wedge^2 V}$ belongs to the class (78) with $N = 3$.

The matter curve for $\wedge^2 V$ is given by $a_3 = 0$. This is because (84) determines three points in each elliptic fiber, and one of the three points approaches the zero section σ as $a_3 \to 0$. This is also because the two points determined by (83) share the same value of x, $-a_0/a_2$, if $a_3 = 0$; two points on an elliptic curve $p_j = (x, y)$ and $p_k = (x, -y)$ are inverse elements of each other in terms of group law on the elliptic curve, that is, $p_j \boxplus p_k = e_0$. The matter curve $\bar{c}_{\wedge^2 V}$, specified by $a_3 = 0$, belongs to a class (79) for $N = 3$.

Since $\wedge^2 V = V^\times \otimes \det V$, and $\det V = \pi_Z^* \mathcal{O}(\pi_{C*}\gamma)$, it is straightforward to obtain the sheaf $\mathcal{F}_{\wedge^2 V}$ on the matter curve $\bar{c}_{\wedge^2 V} = \bar{c}_V$. Applying $\otimes \mathcal{O}(\pm \pi_{C*}\gamma)$ to $\mathcal{F}_{V^\times}$ and $\mathcal{F}_V$,

$$\mathcal{F}_{\wedge^2 V} = \mathcal{O}\left(i^* K_{B_2} + \frac{1}{2} j^* r - j^*\gamma + i^*\pi_{C*}\gamma \right), \tag{85}$$

$$\mathcal{F}_{\wedge^2 V^\times} = \mathcal{O}\left(i^* K_{B_2} + \frac{1}{2} j^* r + j^*\gamma - i^*\pi_{C*}\gamma \right), \tag{86}$$

where $i : \bar{c}_{\wedge^2 V} = \bar{c}_V \hookrightarrow B_2$, and $j : \bar{c}_{\wedge^2 V} = \bar{c}_V \hookrightarrow C_V$. It is thus unnecessary to use the idea presented in section 4 in determining the $\mathcal{F}_{\wedge^2 V}$ for rank-3 bundles V. In the rest of section 5.2, however, we use the idea to reproduce this result, so that we get accustomed to using the idea in practice.

In the fiber E_b of an arbitrary point $b \in \bar{c}_V = \bar{c}_{\wedge^2 V} \subset B_2$, $C_V|_{E_b}$ consists of three points, one in the zero section $p_i = e_0 = \sigma \cdot E_b$ and two others satisfying $p_j \boxplus p_k = e_0$. Thus, the irreducible decomposition (69) becomes

$$C_V \cdot Y = D + \bar{c}_V. \tag{87}$$

The curve D is already a degree-2 cover on $\bar{c}_V = \bar{c}_{\wedge^2 V}$, and we do not need to introduce a covering curve $\tilde{\bar{c}}_{\wedge^2 V}$ for rank-3 bundles V.

Among various components of divisors specifying the rank-1 sheaf $\mathcal{F}_{\wedge^2 V}$ in (77), $\pi_{D*}\gamma$ and $\pi_{D*}(r|_D - R)/2$ can be treated separately. Because of the irreducible decomposition we have seen above,

$$\pi_{D*}\gamma = i^*\pi_{C*}\gamma - j^*\gamma, \tag{88}$$

and hence the γ-dependent part of (85) is reproduced.

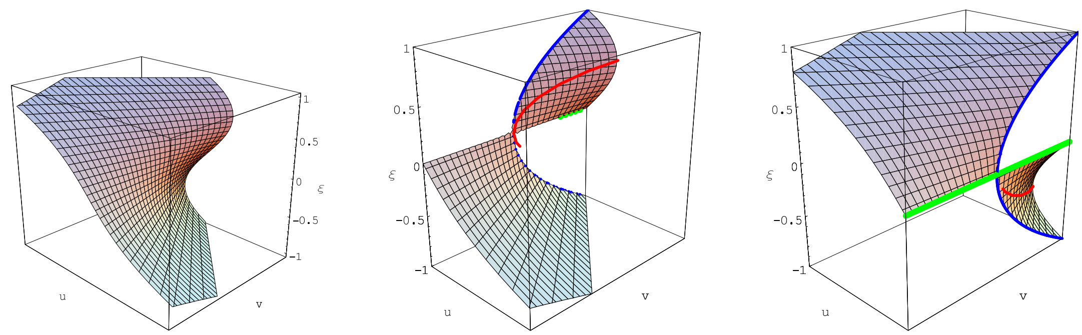
The remaining task is to examine $\pi_{D*}(r|_D - R)/2$. Because the spectral surface C_V is ramified over σ whenever $D \subset C_V$ is on $\bar{c}_{\wedge^2 V}$, we begin with classifying the intersection points of the two divisors r and D in C_V. For rank-3 bundles V, there are two types of r–D intersection points on C_V:

(a) $p_j = p_k = e_0$, where $p_i = e_0$, too,

(b) $p_j = p_k = e'$, where e' denotes one of three points of order two in an elliptic curve E_b (i.e., $e' \boxplus e' = e_0$).

Figure 1 shows the behavior of the spectral surface C_V around a D–r intersection point of type (a). From the figure, one can read off that

$$\deg r|_D = D \cdot r = 2, \qquad \deg R = 1 \tag{89}$$



(i) (ii) (iii)

Figure 1: (i) is a local picture of spectral surface C_V around a point where D and r intersect as (a) in the classification in the text. Among the three local coordinates (ξ, u, v) (x is used as axes label in the figures instead of ξ), (u, v) are for the base 2-fold B_2, and ξ is for the elliptic fiber. $(u, v) = (0, 0)$ corresponds to a type (a) point in $\bar{c}_{\wedge^2 V} = \bar{c}_V$, and $\xi = 0$ to the zero section σ. (ii) and (iii) cut out $u \geq 0$ and $u \leq 0$ parts of (i), so that curves $\bar{c}_V = \bar{c}_{\wedge^2 V}$ (thick green), D (thin blue) and r (thin red) are clearly visible. See the appendix B.2 for more details.

at each type (a) D–r intersection point. Explicit calculation of deg $r|_D$ at type (a) intersection points is found in the appendix B.2. On the other hand, at a type (b) D–r intersection, one can see that

$$\deg r|_D = 1, \qquad \deg R = 1. \tag{90}$$

Therefore, deg $(r|_D - R) = 1$ remains valid at each type (a) D–r intersection points, while deg $(r|_D - R) = 0$ at type (b) D–r intersections. Thus, by denoting collection of the image of all the type (a) D–r intersection points by π_D as $b^{(a)}$, we find that

$$\frac{1}{2}\pi_{D*}(r|_D - R) = \frac{1}{2}b^{(a)}. \tag{91}$$

One can see that (a) and (b) exhaust all the r–D intersection points. First, type (a) intersection points mapped by π_D to $\bar{c}_V = \bar{c}_{\wedge^2 V}$ are characterized by $a_3 = 0$ and $a_2 = 0$. Thus, there are

$$\deg b^{(a)} = (3K_{B_2} + \eta) \cdot (2K_{B_2} + \eta) \tag{92}$$

of them. Type (b) intersection points are characterized by the intersection of the curve D and a locus of order-2 points, σ'. σ' (denoted by σ_2 in [18]) is topologically $\sigma' \sim 3\sigma - 3K_{B_2}$ [25]. Using the topological form of D in (80) for $N = 3$, we find that there are

$$D \cdot \sigma' = [\sigma \cdot (6K_{B_2} + 2\eta) + \pi_Z^* \eta \cdot (3K_{B_2} + \eta)] \cdot (3\sigma - 3\pi_Z^* K_{B_2}) = \sigma \cdot 3\eta \cdot (3K_{B_2} + \eta) \tag{93}$$

type (b) r–D intersection points. Now remembering that deg $r|_D = 2$ at each type (a) intersection point and $= 1$ at type (b) intersection points, it is easy to see that the intersection number

$$\begin{aligned} D \cdot r &= [\sigma \cdot (6K_{B_2} + 2\eta) + \pi_Z^* \eta \cdot (3K_{B_2} + \eta)] \cdot [3\sigma + \pi_Z^*(\eta - K_{B_2})]\,, && (94) \\ &= (3K_{B_2} + \eta) \cdot \sigma \cdot (2\,(2K_{B_2} + \eta) + 3\eta) && (95) \end{aligned}$$

is accounted for by the type (a) and type (b) intersection points. We used $r \sim K_{C_V} - \pi_C^* K_{B_2} \sim N\sigma + \pi_C^*(\eta - K_{B_2})$ for $N = 3$.

Once we show that

$$j^* r = r|_{\bar{c}_V} = b^{(a)}, \tag{96}$$

then (77), (88) and (91) reproduce (85). To see this relation between $r|_{\bar{c}_V}$ and $b^{(a)}$, it is sufficient to count the number of $\bar{c}_V$–r intersection points in C_V. This is because deg $j^* r = 1$ at each type (a) intersection point (as one can see intuitively from Figure 1, or from explicit calculation in the appendix B.2). The intersection number $\bar{c}_V \cdot r$ in C_V is given by

$$\text{deg } j^* r = \bar{c}_V \cdot r = \sigma \cdot (N\sigma + \pi_C^*(\eta - K_{B_2})) = \sigma \cdot ((N-1)K_{B_2} + \eta) = (NK_{B_2} + \eta) \cdot ((N-1)K_{B_2} + \eta) \tag{97}$$

for $N = 3$, and hence is the same as (92). We have finally seen that (77) reproduces (85) properly.

Once the sheaf (and in particular, line bundle) for $\wedge^2 V$ is obtained, its net chirality follows immediately. Using the Riemann–Roch theorem on the matter curve $\bar{c}_{\wedge^2 V} = \bar{c}_V$,

$$\begin{aligned} \chi(\wedge^2 V) &= 1 - g(\bar{c}_V) + \text{deg } \mathcal{F}_{\wedge^2 V}, && (98) \\ &= 1 - g(\bar{c}_V) + \text{deg } \mathcal{F}_{V^\times} + \text{deg } i^* \pi_{C*} \gamma && (99) \\ &= -\chi(V) + (\pi_{C*}\gamma) \cdot (3K_B + \eta). && (100) \end{aligned}$$

This calculation confirms, using only the sheaves on the matter curves, that a consistency relation (375) between $\chi(V)$ and $\chi(\wedge^2 V)$ is satisfied.

5.3 Rank-4 Vector Bundles

Let us now study $R^1 \pi_{Y*} \wedge^2 V$ for rank-4 bundles V. The spectral surface of a rank-4 bundle V is a zero locus of

$$s = a_0(u,v) + a_2(u,v)x + a_3(u,v)y + a_4(u,v)x^2, \tag{101}$$

where (u,v) are local coordinates on the base manifold B_2, and (x,y) describe the elliptic fiber. The matter curve $\bar{c}_V$ for the fundamental representation V is given by $a_4=0$, since one of the solutions becomes $(x,y)=(\infty,\infty)=e_0$. The matter curve for $\wedge^2 V$, $\bar{c}_{\wedge^2 V}$ is determined by the condition that s in (101) factorizes as

$$s=(Ax+B)(Px+Q). \tag{102}$$

Thus, $\bar{c}_{\wedge^2 V}$ denotes the locus $a_3=0$. If s factorized[8] for a point $b\in B_2$, a condition $Ax+B=0$ determines two points in E_b. They are in a relation $p_i=(x,y)$ and $p_j=(x,-y)$, with $x=-B/A$. Thus, $p_i \boxplus p_j = e_0$, and hence $b\in\bar{c}_{\wedge^2 V}$.

Along the matter curve $\bar{c}_{\wedge^2 V}$, there is another pair of points in $C_V|_{E_b}$, $p_k=(x,y)$ and $p_l=(x,-y)$ with $x=-Q/P$ satisfying $p_k\boxplus p_l=e_0$. Thus, all the four points in each fiber of $C_V|_{E_b}$ for $b\in\bar{c}_{\wedge^2 V}$ belong to the component $D\subset C_V\cdot Y$. Thus, the irreducible decomposition (69) becomes

$$C_V\cdot Y = D \tag{103}$$

for rank-4 bundles V. $\pi_D: D\to\bar{c}_{\wedge^2 V}$ is now a degree-4 cover.

The spectral surface $C_{\wedge^2 V}$ forms a double curve along the locus where it intersects with the zero section. One branch corresponds to $p_i\boxplus p_j$ and the other to $p_k\boxplus p_l$. Once Z is blown-up along $\bar{c}_{\wedge^2 V}$ and the double-curve singularity of $C_{\wedge^2 V}$ is resolved, each one of generic points of $\bar{c}_{\wedge^2 V}$ is doubled, one for $p_i\boxplus p_j$ and the other for $p_k\boxplus p_l$, and such points form the covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$. A degree-2 cover $\tilde{\pi}_D: D\to\tilde{\bar{c}}_{\wedge^2 V}$ is defined naturally, but $\nu_{\bar{c}_{\wedge^2 V}}:\tilde{\bar{c}}_{\wedge^2 V}\to\bar{c}_{\wedge^2 V}$ is also a degree-2 cover everywhere[9] on $\bar{c}_{\wedge^2 V}$. This is how $\nu_{\bar{c}_{\wedge^2 V}}\circ\tilde{\pi}_D=\pi_D$ becomes a degree-4 cover.

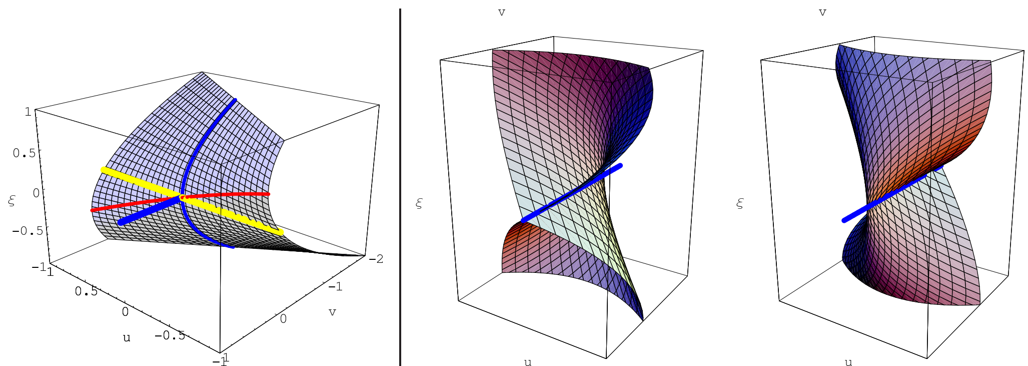
At some special points on the matter curve $\bar{c}_{\wedge^2 V}$, $(Ax+B)=0$ and $(Px+Q)=0$ determine the same pair of points in the fiber. This happens when

$$R^{(4)} := a_2^2-4a_4a_0=(AQ-BP)^2=0 \tag{104}$$

on $\bar{c}_{\wedge^2 V}$. One can further see (in the appendix B.3) that $p_i\boxplus p_j$ and $p_k\boxplus p_l$ are interchanged as a result of monodromy around a zero point of $R^{(4)}$ on a complex curve $\bar{c}_{\wedge^2 V}$. Thus, the covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$ is ramified on $\bar{c}_{\wedge^2 V}$ over zero locus of $R^{(4)}$. We see in the appendix B.3 that $C_{\wedge^2 V}$ develops a codimension-2 singularity at a zero point of $R^{(4)}$. This codimension-2

[8]The factorization of s means that the structure group of the bundle—one that is read out from the spectral surface—is reduced from SU(4) to SU(2) × SU(2). Thus, the commutant of this "structure group" is enhanced from SO(10) to SO(12). This enhanced symmetry determines the form of enhanced singularity along the matter curve $\bar{c}_{\wedge^2 V}$.

[9] The topological class of $\bar{c}_{\wedge^2 V}\in|3K_{B_2}+\eta|$ is different from a naive expectation (79) for $N=2$ by a factor of two, because of this doubling was not taken into account there.



C_V at type (a) $\quad$ $C_{\wedge^2 V}$ at type (c) $\quad$ $C_{\wedge^2 V}$ at type (c)

Figure 2: Coordinates (u, v) describe base manifold B_2, and ξ the fiber direction, with a zero section corresponding to $\xi = 0$. The left panel shows a local picture of C_V for a rank-4 vector bundle around type (a) points on $\bar{c}_{\wedge^2 V}$. Local coordinates (u, v) on the base is chosen so that $a_3 = u$ and $a_4 = v$. $\bar{c}_V$ (thick yellow) and $\bar{c}_{\wedge^2 V}$ (thick blue) in the zero section are given by $a_4 = 0$ and $a_3 = 0$, respectively. Curves D and r on C_V are also shown by thin blue and thin red curves, respectively. Two right panels show local pictures of $C_{\wedge^2 V}$ around type (c) points viewed from opposite directions. Local coordinates are chosen so that $a_3 = u$ (a direction transverse to $\bar{c}_{\wedge^2 V}$), and $a_2^2 - 4a_0 a_4 = v$ on the base manifold. The matter curve $\bar{c}_{\wedge^2 V}$ (thick blue) is a double curve in $C_{\wedge^2 V}$, and a more complicated singularity—called a pinch point—is developed at a type (c) point.

singularity cannot be avoided generically on the matter curve $\bar{c}_{\wedge^2 V}$. We introduce a divisor

$$b^{(c)} := \mathrm{div} R^{(4)} \tag{105}$$

on the matter curve. There are

$$\deg b^{(c)} = (3K_{B_2} + \eta) \cdot (4K_{B_2} + 2\eta) \tag{106}$$

such codimension-2 singularities of $C_{\wedge^2 V}$, which are also the branch points of the degree-2 cover $\nu_{\bar{c}_{\wedge^2 V}} : \tilde{\bar{c}}_{\wedge^2 V} \to \bar{c}_{\wedge^2 V}$.

Apart from these isolated points on $\bar{c}_{\wedge^2 V}$, the idea presented in section 4 can be used to determine the sheaf $\widetilde{\mathcal{F}}_{\wedge^2 V}$ on $\tilde{\bar{c}}_{\wedge^2 V}$. $\tilde{\pi}_D : D \to \tilde{\bar{c}}_{\wedge^2 V}$ determines a divisor $\tilde{\pi}_{D*}\gamma$ on the covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$. The curve D is ramified over $\tilde{\bar{c}}_{\wedge^2 V}$ at two types of points,

(a) $p_i = p_j = e_0$,

(b) $p_i = p_j = e'$.

Thus, there is a potential contribution $\tilde{\pi}_{D*}(r|_D - R)/2$ to the divisor determining $\widetilde{\mathcal{F}}_{\wedge^2 V}$. It turns out, however, that deg $(r|_D - R) = 0$ at each one of type (a) or type (b) points. As we will see shortly, the r–D intersection points of type (a) and type (b) account for all the r–D intersection that are not in the fiber of the zero locus of $R^{(4)}$. Thus, there is no contribution to the divisor from $\tilde{\pi}_{D*}(r|_D - R)/2$ on $\tilde{\bar{c}}_{\wedge^2 V}$ away from the zero locus of $R^{(4)}$, and

$$\widetilde{\mathcal{F}}_{\wedge^2 V} = \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}(\tilde{\pi}_{D*}\gamma). \tag{107}$$

The remaining r–D intersection points on C_V come from type (c) intersection points

(c) $p_i = p_k =: p_+$ and $p_j = p_l =: p_-$.

These points are in the fiber of the zero locus of $R^{(4)}$. The number of all those types of r–D intersection points are given by

$$\begin{aligned}
\#(\mathrm{a}) &= D \cdot \sigma = (4K_{B_2} + \eta) \cdot (3K_{B_2} + \eta), && (108)\\
\#(\mathrm{b}) &= D \cdot \sigma' = D \cdot 3(\sigma - K_{B_2}), && (109)\\
\#(\mathrm{c}) &= 2 \times \deg b^{(c)} = 2\, \bar{c}_{\wedge^2 V} \cdot (4K_{B_2} + 2\eta). && (110)
\end{aligned}$$

All these r–D intersection points contribute to the intersection number $D \cdot r$ with unit multiplicity. We can now see that $D \cdot r$ is accounted for by these intersection points:

$$\begin{aligned}
D \cdot r &= D \cdot (4\sigma + \eta - K_{B_2}), && (111)\\
&= D \cdot \sigma + D \cdot (3\sigma - 3K_{B_2}) + D \cdot (2K_{B_2} + \eta), && (112)\\
&= \#(\mathrm{a}) + \#(\mathrm{b}) + \#(\mathrm{c}). && (113)
\end{aligned}$$

Let us now study the structure of $R^1\pi_{Z*}\wedge^2 V$ in a local neighborhood of a zero point of $R^{(4)}$. We assume[10] that $R^1\pi_{Z*}\wedge^2 V$ is written as $i_{\wedge^2 V*}\mathcal{F}$ for some sheaf $\mathcal{F}$ on $\bar{c}_{\wedge^2 V}$. Then,

$$\begin{aligned}\mathcal{F} &\cong i^*_{\wedge^2 V*} R^1\pi_{Z*}\wedge^2 V &(114)\\ &\cong R^1\pi_{Y*}(\wedge^2 V|_Y), &(115)\end{aligned}$$

where we used the base change formula in the second isomorphism.

Let $b \in \bar{c}_{\wedge^2 V}$ be a zero locus of $R^{(4)}$. Locally (in the analytic topology) around b, the curve D is decomposed into a disjoint union of D_+ and D_-. We consider the following diagram,

$$\begin{tikzcd}
 & \tilde{D}_\dagger \times_{\tilde{\bar{c}}_{\wedge^2 V}} \tilde{Y} \arrow[dl, "p_1"'] \arrow[drr, "p_2"] \arrow[rr, "\nu_{\tilde{D}_\dagger}"] & & D \times_{\bar{c}_{\wedge^2 V}} Y \arrow[dr, "p_2"] & \\
\tilde{D}_\dagger \arrow[dr, "\pi_{\tilde{D}}"'] & & & \tilde{Y} \arrow[r, "\nu_Y"] \arrow[dll, "\pi_{\tilde{Y}}"] & Y \arrow[dl, "\pi_Y"] \\
 & \tilde{\bar{c}}_{\wedge^2 V} \arrow[rr, "\nu_{\bar{c}_{\wedge^2 V}}"'] & & \bar{c}_{\wedge^2 V}. &
\end{tikzcd} \tag{116}$$

Here $\tilde{Y} = \tilde{\bar{c}}_{\wedge^2 V} \times_{\bar{c}_{\wedge^2 V}} Y$ (as we have already introduced in section 4), and $\tilde{D}_\dagger = \tilde{\bar{c}}_{\wedge^2 V} \times_{\bar{c}_{\wedge^2 V}} D$. We have the decompositions,

$$\tilde{D}_\dagger = \tilde{D}_+ \coprod \tilde{D}_-, \quad \tilde{D}_\pm = \tilde{D}^{(1)}_\pm \cup \tilde{D}^{(2)}_\pm,$$

where $\tilde{D}_\pm = \tilde{\bar{c}}_{\wedge^2 V} \times_{\bar{c}_{\wedge^2 V}} D_\pm$ and $\tilde{D}^{(i)}_\pm$ for $i = 1, 2$ are irreducible components of $\tilde{D}_\pm$. Note that each $\tilde{D}^{(i)}_\pm$ is a section of $\pi_{\tilde{Y}}$, and $\tilde{D}^{(1)}_\pm$, $\tilde{D}^{(2)}_\pm$ intersect at one point transversally, say $\tilde{p}_\pm \in \tilde{D}^{(1)}_\pm \cap \tilde{D}^{(2)}_\pm$. Moreover we may assume $\tilde{D}^{(1)}_+ \boxplus \tilde{D}^{(2)}_-$ and $\tilde{D}^{(1)}_- \boxplus \tilde{D}^{(2)}_+$ are zero sections of $\pi_{\tilde{Y}}$. See Figure 3. Since $\nu_{\bar{c}_{\wedge^2 V}}$ is a Galois cover with Galois group $G = \mathbb{Z}/2\mathbb{Z}$, we have

$$R^1\pi_{Y_*}(\wedge^2 V|_Y) \cong \left(\nu_{\bar{c}_{\wedge^2 V}*}\nu^*_{\bar{c}_{\wedge^2 V}} R^1\pi_{Y_*}(\wedge^2 V|_Y)\right)^G.$$

[10] It is not obvious whether $R^1\pi_{Z*}\wedge^2 V$ is represented as $i_{\wedge^2 V*}\mathcal{F}$ as a sheaf of $\mathcal{O}_{B_2}$-module, although the support of $R^1\pi_{Z*}\wedge^2 V$ is $\bar{c}_{\wedge^2 V}$. See the appendix A for more. To show that this is the case, we need to see that the ideal sheaf of $\bar{c}_{\wedge^2 V}$ acts trivially on $R^1\pi_{Z*}\wedge^2 V$. We have seen in the appendix A and section 4 that this is true for generic points on the matter curve $\bar{c}_{\wedge^2 V}$. Double-curve singularity on $C_{\wedge^2 V}$ does not pose a problem. However, we have not shown this in a neighborhood containing a zero locus of $R^{(c)}$.
As we see in the appendix B.3, $C_{\wedge^2 V}$ approaches the zero section as either $\xi \sim \pm(w_+ - w_-)$ or $\xi \sim \pm(w_+ + w_-)$. Since the normal coordinate of $\bar{c}_{\wedge^2 V}$ is $u \propto (w_+ + w_-)(w_+ - w_-)$ around the zero locus of $R^{(4)}$, it sounds quite reasonable that the normal coordinate acts trivially on the generator of $R^1\pi_{Z*}\wedge^2 V$, just like the normal coordinate does in the case discussed in the appendix A. But, we have not completed a proof, and we just leave this as an assumption.

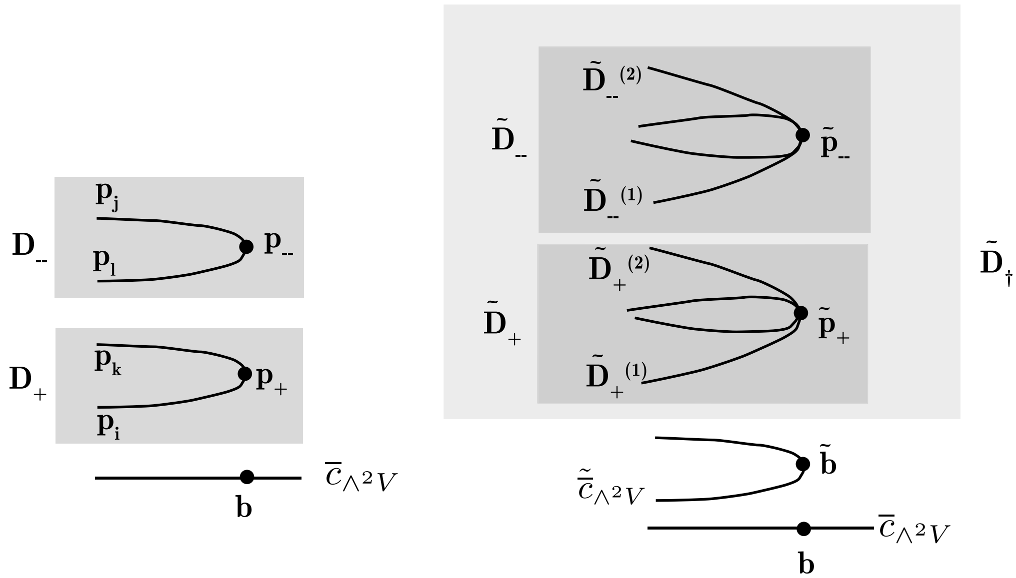


Figure 3: A schematic picture showing relations between various curves and points that are used in the text.

By the base change formula, we have

$$\nu^*_{\tilde{\bar{c}}_{\wedge^2 V}} R^1\pi_{Y_*}(\wedge^2 V|_Y) \cong R^1\pi_{\tilde{Y}*}(\wedge^2 \nu_Y^*(V|_Y)), \tag{117}$$

and

$$\nu_Y^* V|_Y \cong p_{2*}(p_1^*\mathcal{N}_V|_{\tilde{D}_\dagger} \otimes \mathcal{P}_{\tilde{\bar{c}}_{\wedge^2 V}}) \tag{118}$$

$$\cong p_{2*}(p_1^*\mathcal{N}_V|_{\tilde{D}_+} \otimes \mathcal{P}_{\tilde{\bar{c}}_{\wedge^2 V}}) \oplus p_{2*}(p_1^*\mathcal{N}_V|_{\tilde{D}_-} \otimes \mathcal{P}_{\tilde{\bar{c}}_{\wedge^2 V}}), \tag{119}$$

Here $\mathcal{P}_{\tilde{\bar{c}}_{\wedge^2 V}}$, $\mathcal{N}_V|_{\tilde{D}_*}$ for $* = \dagger, \pm$ are pullbacks of $\mathcal{P}_B$, $N_V|_{\tilde{D}_*}$ via $\nu_{\tilde{D}_\dagger}$ and the second projection $\tilde{D}_* \to D$ respectively. Let us set $W_\pm = p_{2*}(p_1^*\mathcal{N}_V|_{\tilde{D}_\pm} \otimes \mathcal{P}_{\tilde{\bar{c}}_{\wedge^2 V}})$. We have

$$\nu^*_{\tilde{\bar{c}}_{\wedge^2 V}} R^1\pi_{Y_*}(\wedge^2 V|_Y) \cong R^1\pi_{\tilde{Y}*}(W_+ \otimes W_-). \tag{120}$$

It is useful in calculating $R^1\pi_{\tilde{Y}*}(W_+ \otimes W_-)$ to note that

$$0 \to \mathcal{O}_{\tilde{D}_\pm} \to \mathcal{O}_{\tilde{D}^{(1)}_\pm} \oplus \mathcal{O}_{\tilde{D}^{(2)}_\pm} \to \mathcal{O}_{\tilde{p}_\pm} \to 0 \tag{121}$$

are exact. Applying $\otimes \mathcal{N}_V|_{\tilde{D}_\dagger}$ and Fourier-Mukai transforms, we obtain the exact sequences,

$$0 \to W_\pm \to W^{(1)}_\pm \oplus W^{(2)}_\pm \to W^{(0)}_\pm \to 0. \tag{122}$$

Here $W_{\pm}^{(*)}$ for $*=0,1,2$ are Fourier–Mukai transforms,

$$W_{\pm}^{(*)} = p_{2*}(p_1^*\mathcal{N}_V|_{\tilde{D}_{\pm}^{(*)}} \otimes \mathcal{P}_{\tilde{\bar{c}}_{\wedge^2 V}}), \tag{123}$$

where $\tilde{D}_{\pm}^{(0)} = \tilde{p}_{\pm}$. Note that $W_{\pm}^{(*)}$ for $*=1,2$ is a line bundle on Y and $W_{\pm}^{(0)}$ is a line bundle on the fiber $\tilde{Y}_b = \pi_{\tilde{Y}}^{-1}\nu_{\bar{c}_{\wedge^2 V}}^{-1}(b)$.

Applying $\otimes W_-^{(1)}$ to the above sequence yields the exact sequence,

$$0 \to W_+ \otimes W_-^{(1)} \to (W_+^{(1)} \otimes W_-^{(1)}) \oplus (W_+^{(2)} \otimes W_-^{(1)}) \to W_+^{(0)} \otimes W_-^{(1)} \to 0. \tag{124}$$

$W_+^{(0)} \otimes W_-^{(1)}$ is a trivial line bundle on $\tilde{Y}_b$, and $W_+^{(2)} \otimes W_-^{(1)}$ is also a line bundle that is trivial in the elliptic fiber direction. Thus, by applying $R^i\pi_{\tilde{Y}*}$, we have the long exact sequence,

$$\begin{aligned} 0 &\longrightarrow R^0\pi_{\tilde{Y}*}(W_+ \otimes W_-^{(1)}) \longrightarrow \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}\left(\tilde{b} + \tilde{\pi}_{D*}\gamma\right) \twoheadrightarrow \mathcal{O}_{\tilde{b}} \\ &\longrightarrow R^1\pi_{\tilde{Y}*}(W_+ \otimes W_-^{(1)}) \longrightarrow \mathcal{O}_{\tilde{b}} \oplus \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}\left(\tilde{b} + \tilde{\pi}_{D*}\gamma + \tilde{\imath}_{\wedge^2 V}^* K_{B_2}\right) \longrightarrow \mathcal{O}_{\tilde{b}} \longrightarrow 0, \end{aligned} \tag{125}$$

where $\tilde{b} := \nu_{\bar{c}_{\wedge^2 V}}^{-1}(b)$, $\tilde{\imath}_{\wedge^2 V} := i_{\wedge^2 V} \circ \nu_{\bar{c}_{\wedge^2 V}}$, and we used

$$R^0\pi_{\tilde{Y}*}(W_+^{(2)} \otimes W_-^{(1)}) \cong \tilde{\pi}_{D_+*}(\mathcal{N}_V|_{D_+}) \otimes \tilde{\pi}_{D_-*}(\mathcal{N}_V|_{D_-}) \cong \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}(\tilde{b} + \tilde{\pi}_{D*}\gamma), \tag{126}$$

$$R^1\pi_{\tilde{Y}*}(W_+^{(2)} \otimes W_-^{(1)}) \cong \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}(\tilde{b} + \tilde{\pi}_{D*}\gamma) \otimes \mathcal{L}_H^{-1} \cong \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}(\tilde{b} + \tilde{\pi}_{D*}\gamma + \tilde{\imath}_{\wedge^2 V}^* K_{B_2}); \tag{127}$$

the ramification divisor r on C_V intersects with $D_{\pm}$ at $p_{\pm}$, and $\tilde{\pi}_{D_{\pm}} = \tilde{\pi}_D|_{D_{\pm}} : D_{\pm} \to \tilde{\bar{c}}_{\wedge^2 V}$ maps $p_{\pm}$ to $\tilde{b} \in \tilde{\bar{c}}_{\wedge^2 V}$. This is why we have a divisor $\tilde{b}$ in (126). We thus conclude that

$$R^0\pi_{\tilde{Y}*}(W_+ \otimes W_-^{(1)}) \cong \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}(\tilde{\pi}_{D*}\gamma), \tag{128}$$

$$R^1\pi_{\tilde{Y}*}(W_+ \otimes W_-^{(1)}) \cong \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}(\tilde{b} + \tilde{\pi}_{D*}\gamma + \tilde{\imath}_{\wedge^2 V}^* K_{B_2}). \tag{129}$$

By the same argument, we also have the same results for $R^i\pi_{\tilde{Y}*}(W_+ \otimes W_-^{(2)})$ $(i=0,1)$.

Finally we have the exact sequence,

$$0 \to W_+ \otimes W_- \to (W_+ \otimes W_-^{(1)}) \oplus (W_+ \otimes W_-^{(2)}) \to W_+ \otimes W_-^0 \to 0. \tag{130}$$

Note that $W_+ \otimes W_-^0$ is a rank two degree-zero sheaf on an elliptic curve $\tilde{Y}_b$ given in [26]. Thus, we have the associated long exact sequence,

$$\begin{aligned} &\mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}(\tilde{\pi}_{D*}\gamma) \oplus \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}(\tilde{\pi}_{D*}\gamma) \twoheadrightarrow \mathcal{O}_{\tilde{b}} \longrightarrow R^1\pi_{\tilde{Y}*}(W_+ \otimes W_-) \\ &\longrightarrow \oplus^2 \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}(\tilde{b} + \tilde{\pi}_{D*}\gamma + \tilde{\imath}_{\wedge^2 V}^* K_{B_2}) \longrightarrow \mathcal{O}_{\tilde{b}} \longrightarrow 0. \end{aligned} \tag{131}$$

Therefore, we obtain

$$\begin{aligned} R^1\pi_{\tilde{Y}*}(W_+\otimes W_-) &\cong \mathrm{Ker}\left(\mathcal{O}(\tilde{b}+\tilde{\pi}_{D*}\gamma+\tilde{\imath}^*_{\wedge^2 V}K_{B_2})\oplus\mathcal{O}(\tilde{b}+\tilde{\pi}_{D*}\gamma+\tilde{\imath}^*_{\wedge^2 V}K_{B_2})\to\mathcal{O}_{\tilde{b}}\right), \\ &= \left\{(f,g)|f,g\in\mathcal{O}(\tilde{b}+\tilde{\pi}_{D*}\gamma+\tilde{\imath}^*_{\wedge^2 V}K_{B_2}),\quad f|_{\tilde{b}}=g|_{\tilde{b}}\right\} \end{aligned} \tag{132}$$

Under the above isomorphism, we can easily see that the action of G on $\nu_{\bar{c}_{\wedge^2 V}*}R^1\pi_{\tilde{Y}*}(W_+\otimes W_-)$ is given by $(f(\tilde{u}),g(\tilde{u}))\mapsto(g(-\tilde{u}),f(-\tilde{u}))$, where $\tilde{u}$ is the local coordinate of $\tilde{\bar{c}}_{\wedge^2 V}$ around $\tilde{b}$. Hence we have

$$\mathcal{F}_{\wedge^2 V}\cong\left(\nu_{\bar{c}_{\wedge^2 V}*}R^1\pi_{\tilde{Y}*}(W_+\otimes W_-)\right)^G \tag{133}$$

$$\cong\nu_{\bar{c}_{\wedge^2 V}*}\mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}}(\tilde{b}+\tilde{\pi}_{D*}\gamma+\tilde{\imath}^*_{\wedge^2 V}K_{B_2}). \tag{134}$$

Therefore, after making the assumption discussed in footnote 10, we find that $\mathcal{F}_{\wedge^2 V}$ on $\bar{c}_{\wedge^2 V}$ is given by a pushforward of a locally free rank-1 sheaf[11]

$$\widetilde{\mathcal{F}}_{\wedge^2 V}=\mathcal{O}\left(\tilde{\imath}^*_{\wedge^2 V}K_{B_2}+\tilde{b}^{(c)}+\tilde{\pi}_{D*}\gamma\right) \tag{135}$$

on $\tilde{\bar{c}}_{\wedge^2 V}$ via $\nu_{\bar{c}_{\wedge^2 V}}$ everywhere on $\bar{c}_{\wedge^2 V}$. Here, $\tilde{b}^{(c)}$ denotes a divisor $\nu^{-1}_{\bar{c}_{\wedge^2 V}}b^{(c)}$, collecting all the points that we have denoted as $\tilde{b}$ up to now.

Matter chiral multiplets from the $\wedge^2 V$ bundle are now identified with

$$H^1(Z;\wedge^2 V)\simeq H^0(\tilde{\bar{c}}_{\wedge^2 V};\widetilde{\mathcal{F}}_{\wedge^2 V})\simeq H^0(\bar{c}_{\wedge^2 V};\mathcal{F}_{\wedge^2 V}). \tag{136}$$

$\mathcal{F}_{\wedge^2 V}=\nu_{\bar{c}_{\wedge^2 V}*}\widetilde{\mathcal{F}}_{\wedge^2 V}$ is a locally-free rank-2 sheaf (rank-2 vector bundle) on $\bar{c}_{\wedge^2 V}$.

The genus of the covering matter curve is given by

$$g(\tilde{\bar{c}}_{\wedge^2 V})=1+2(g(\bar{c}_{\wedge^2 V})-1)+\frac{1}{2}\mathrm{deg}\ b^{(c)}, \tag{137}$$

since $\nu_{\bar{c}_{\wedge^2 V}}:\tilde{\bar{c}}_{\wedge^2 V}\to\bar{c}_{\wedge^2 V}$ is a degree-2 cover with $(1/2)\mathrm{deg}\ b^{(c)}$ branch cuts. Thus, it is also expressed as

$$g(\tilde{\bar{c}}_{\wedge^2 V})=1+(3K_{B_2}+\eta)\cdot(4K_{B_2}+\eta)+\frac{1}{2}(3K_{B_2}+\eta)\cdot(4K_{B_2}+2\eta), \tag{138}$$

[11] This result follows immediately from (77), had we made a stronger assumption that there exists a *locally free rank-1* sheaf $\widetilde{\mathcal{N}}_{\wedge^2 V}$ on $\widetilde{C}_{\wedge^2 V}$ such that $\mathcal{N}_{\wedge^2 V}:=\nu_{\bar{c}_{\wedge^2 V}*}\widetilde{\mathcal{N}}_{\wedge^2 V}$ is used in (22) and (22) is satisfied as a sheaf of $\mathcal{O}_Z$-module. We opted to adopt a weaker assumption in this article, and provided a derivation after (115), instead.

and it follows that

$$\deg K_{\tilde{\bar{c}}_{\wedge^2 V}} = 2 \times (3K_{B_2} + \eta) \cdot (6K_{B_2} + 2\eta). \tag{139}$$

On the other hand, one can also calculate the following independently:

$$\begin{aligned} \deg \left(\tilde{\imath}^*_{\wedge^2 V} K_{B_2} + \tilde{b}^{(c)}\right) &= 2(3K_{B_2} + \eta) \cdot K_{B_2} + (3K_{B_2} + \eta) \cdot (4K_{B_2} + 2\eta), \\ &= (3K_{B_2} + \eta) \cdot (6K_{B_2} + 2\eta) = \frac{1}{2} \deg K_{\tilde{\bar{c}}_{\wedge^2 V}}. \end{aligned} \tag{140}$$

Because of this non-trivial relation between the genus of the covering curve and the degree of the divisor above, we obtain through Riemann–Roch theorem[12] that

$$\begin{aligned} \chi(\wedge^2 V) &= \chi(\tilde{\bar{c}}_{\wedge^2 V}; \widetilde{\mathcal{F}}_{\wedge^2 V}) = (1 - g(\tilde{\bar{c}}_{\wedge^2 V})) + \deg \left(\tilde{\imath}^*_{\wedge^2 V} K_{B_2} + \tilde{b}^{(c)}\right) + \int_{\tilde{\bar{c}}_{\wedge^2 V}} \tilde{\pi}_{D*}\gamma \\ &= \int_{\tilde{\bar{c}}_{\wedge^2 V}} \tilde{\pi}_{D*}\gamma = \int_{\bar{c}_{\wedge^2 V}} \pi_{D*}\gamma = (3K_{B_2} + \eta) \cdot \pi_{C*}\gamma. \end{aligned} \tag{143}$$

This chirality formula in terms of (covering) matter curve and γ was rather anticipated from the beginning. We know that $\chi(\wedge^2 V) = -\chi(\wedge^2 V^\times)$, and the difference between V and $V^\times$ comes from changing the sign of γ. For an SU(4) bundle V, $\wedge^2 V \simeq \wedge^2 V^\times$ and the net chirality should vanish. We can confirm this in the formula above, because $\pi_{C*}\gamma = 0$ for an SU(4) bundle V. For a U(4) bundle V, its chirality formula (143) agrees with (375) in the appendix that is obtained without calculating direct images. All these consistency checks give us confidence that the locally free rank-1 sheaf (135) provides the right description for the matter multiplets from $\wedge^2 V$.

5.4 Rank-5 Vector Bundles

Spectral surface C_V of rank-5 bundle V is given by

$$s = a_0(u,v) + a_2(u,v)x + a_3(u,v)y + a_4(u,v)x^2 + a_5(u,v)xy = 0; \tag{144}$$

[12]The same result is obtained by applying the Riemann–Roch theorem to $\chi(\bar{c}_{\wedge^2 V}; \mathcal{F}_{\wedge^2 V})$. One needs to use

$$c_1(\mathcal{F}_{\wedge^2 V}) = 2 i^*_{\wedge^2 V} K_{B_2} + \frac{1}{2} b^{(c)} + \pi_{D*}\gamma, \tag{141}$$

and a relation analogous to (140)

$$\deg \left(2 i^*_{\wedge^2 V} K_{B_2} + \frac{1}{2} b^{(c)}\right) = \deg K_{\bar{c}_{\wedge^2 V}}. \tag{142}$$

(u, v) are local coordinates of a base 2-fold B_2, and a_r ($r = 0, 2, 3, 4, 5$) are sections of $\mathcal{O}(rK_{B_2} + \eta)$. The matter curve of the fundamental representation $\bar{c}_V = C_V \cdot \sigma$ is given by the zero locus of a_5, and hence belong to a topological class $|5K_{B_2} + \eta|$. The matter curve of $\wedge^2 V$ is determined by requiring that the defining equation of the spectral surface factorizes[13] locally as

$$s = (Ax + B)(Py + Qx + R). \tag{145}$$

Among the five points $\{p_i, p_j, p_k, p_l, p_m\}$ satisfying (145) in a given elliptic fiber, two points $p_{i,j}$ satisfying $(Ax + B) = 0$ satisfy $p_i \boxplus p_j = e_0$, as we discussed before in section 5.3. One can see that this factorization condition is equivalent to

$$P^{(5)} := a_0 a_5^2 - a_2 a_3 a_5 + a_4 a_3^2 = 0 \tag{146}$$

which was derived in [12]. Since the left-hand side is a section of $\mathcal{O}(10K_{B_2} + 3\eta)$, $\bar{c}_{\wedge^2 V}$ belongs to a class $|10K_{B_2} + 3\eta|$, which corresponds to the $N = 5$ case of (79).

The two matter curves $\bar{c}_V$ and $\bar{c}_{\wedge^2 V}$ intersect in B_2 in general. There are two different types of intersection:

(a) $a_5 = 0$ and $a_4 = 0$, and hence $P^{(5)} = 0$,

(d) $a_5 = 0$ and $a_3 = 0$, and hence $P^{(5)} = 0$.

The two curves intersect with multiplicity 1 at any type (a) intersection points, and with multiplicity 2 at any type (d) intersection points. This is a complete classification of the intersection points of the two matter curves, because

$$\begin{aligned} \#(\mathrm{a}) + 2 \times \#(\mathrm{d}) &= (5K_{B_2} + \eta) \cdot (4K_{B_2} + \eta) + 2 \times (5K_{B_2} + \eta) \cdot (3K_{B_2} + \eta), \\ &= (5K_{B_2} + \eta) \cdot (10K_{B_2} + 3\eta) = \bar{c}_V \cdot \bar{c}_{\wedge^2 V} \end{aligned} \tag{147}$$

accounts for all the contributions to the intersection number. At the type (a) intersection points, five points in C_V become $p_i = p_j = e_0$ and three general points. At the type (d) intersection points, they become $p_m = e_0$, $p_i \boxplus p_j = e_0$ and $p_k \boxplus p_l = e_0$.

The explicit form of $P^{(5)}$ reveals that $\bar{c}_{\wedge^2 V}$ forms a double point at each type (d) intersection point. This is where a double-curve locus $p_i \boxplus p_j = p_k \boxplus p_l$ of $C_{\wedge^2 V}$ intersects with the zero section. The projection $\pi_D : D \to \bar{c}_{\wedge^2 V}$ is a degree-2 cover at generic points on

[13] "The structure group of the spectral surface" is reduced from SU(5) to SU(2)×SU(3), and the commutant within E_8 enhanced from $\mathrm{SU}(5)_{\mathrm{GUT}}$ to SU(6). As we have already noted in section 5.3, it is this "structure group of the spectral surface" that determines the enhanced singularity along the matter curve in F-theory dual description.

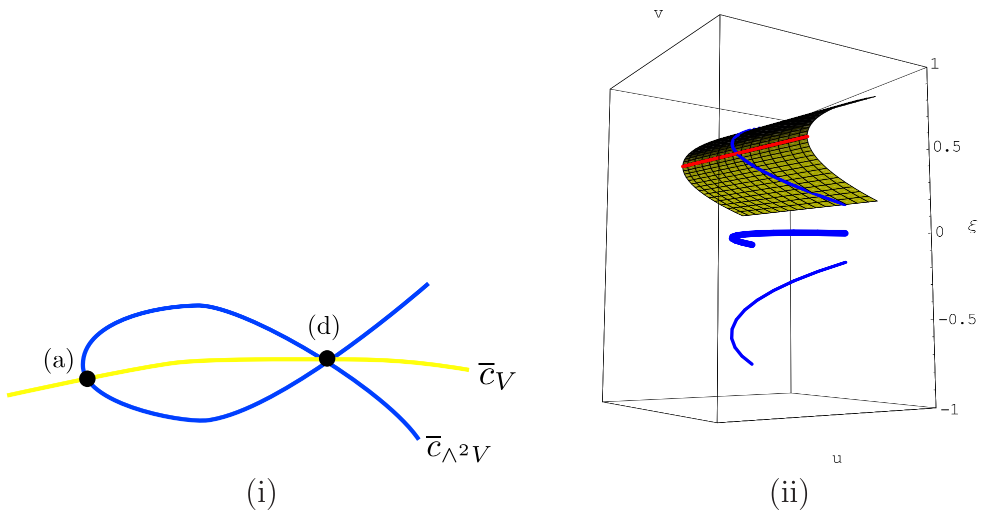


Figure 4: The left panel (i) shows how the two matter curves $\bar{c}_V$ (thick yellow) and $\bar{c}_{\wedge^2 V}$ (thick blue) intersect in the zero section for cases with rank-5 bundles V. The right panel (ii) shows geometry of a curve D (thin blue) and $\bar{c}_{\wedge^2 V}$ (thick blue) associated with a type (c1) point. Only a degree-2 part of degree-5 spectral cover surface C_V is shown here. C_V is a ramified cover over B_2 along a ramification divisor r (thin red), but D is not a ramified cover over $\bar{c}_{\wedge^2 V}$.

$\bar{c}_{\wedge^2 V}$, but the four points $p_{i,j,k,l}$ map to a type (d) point on $\bar{c}_{\wedge^2 V}$. The covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$ is obtained by blowing up the double points of the matter curve $\bar{c}_{\wedge^2 V}$, and the map $\tilde{\pi}_D : D \longrightarrow \tilde{\bar{c}}_{\wedge^2 V}$ becomes a degree-2 cover. The idea of section 4 is applied, and a locally free rank-1 sheaf $\widetilde{\mathcal{F}}_{\wedge^2 V}$ on the covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$ is obtained. The sheaf $\mathcal{F}_{\wedge^2 V}$ on the matter curve $\bar{c}_{\wedge^2 V}$ is locally free and has rank 1 in a neighborhood of a generic point, but it is not locally free around the type (d) intersection points as the rank of the fiber jumps from 1 to 2.

We would like to better understand the locally free rank-1 sheaf (77), by studying $\pi_{D*}(r|_D - R)$ in more detail. As we have discussed in section 5.2, C_V is ramified over σ whenever D is over $\tilde{\bar{c}}_{\wedge^2 V}$. Thus, $\mathrm{supp}(r|_D - R) \subset \mathrm{supp}\, r|_D$ on D, and we begin with a classification of the D–r intersection points on C_V. This must include all the intersection points:

- (a) $p_i = p_j = e_0$ $(p_i, p_j \in D)$; $\qquad \#(\text{a}) = D \cdot \sigma$
- (b) $p_i = p_j = e'$ $(p_i, p_j \in D)$; $\qquad \#(\text{b}) = D \cdot \sigma'$,
- (c) others.

The type (a) and type (b) intersection points exhaust all the cases where $\deg R \neq 0$. The

type (a) D–r intersection points are also the type (a) $\bar{c}_V$–$\bar{c}_{\wedge^2 V}$ intersection points. As we have seen before for rank-4 bundles, $\deg\, r|_D = \deg\, R = 1$ at both type (a) and type (b) D–r intersections points, and no contribution to $\tilde{\pi}_{D*}(r|_D - R)/2$ arises from these points.[14]

The images of the remaining D–r intersections points—called type (c) points—via $\tilde{\pi}_{D*}$ define a divisor $\tilde{b}^{(c)}$ on $\tilde{\bar{c}}_{\wedge^2 V}$. Since $\deg\, R = 0$ at the type (c) D–r intersection points, $\tilde{\pi}_{D*}(r|_D - R)/2$ becomes $\tilde{b}^{(c)}/2$. From the definition of the type (c) points, the number of such points is given by

$$\deg\, \tilde{b}^{(c)} = D\cdot r - D\cdot\sigma - D\cdot\sigma' = D\cdot((N-4)\sigma + 2K_{B_2} + \eta), \tag{148}$$

with $N = 5$.

This paragraph was modified in version 4: The type (c) D–r intersection points are characterized as follows. In the right panel of Figure 4, the two curves D and r intersect on the spectral surface C_V, but $D \to \bar{c}_{\wedge^2 V}$ is not a ramified cover. We count the number of such D–r intersection points in the following, and show that the number agrees with (148). Let us denote the point at this type of D–r intersection as p_i, and the other point on D as p_j; $p_i \boxplus p_j = e_0$ by definition. On an elliptic fiber E_b that has such p_i and p_j in it, the defining equation of the spectral surface (144) has p_i as a zero of order two. Assuming that $a_5 \neq 0$, the points p_i and p_j correspond to $(x_*, \pm y_*)$ with $x_* = -a_3/a_5$ and $y_*^2 = x_*^3 + f_0 x_* + g_0$. Either p_i or p_j being a zero of order two of (144) means that

$$\frac{ds}{dx} = (a_2 + 2a_4 x + a_5 y) + \frac{dy}{dx}(a_3 + a_5 x) \tag{149}$$

vanishes at either p_i or p_j. The second term always vanishes for $x_* = -a_3/a_5$. Thus, the condition becomes $(a_2 + 2a_4 x_*)^2 - a_5^2 y_*^2 = 0$. Writing x_* and y_* in terms of $a_{0,2,3,4,5}$ and f_0 and g_0, we find that this condition is equivalent to

$$R^{(5)} := \left(a_2 - \frac{2a_4 a_3}{a_5}\right)^2 + a_5^2\left(\left(\frac{a_3}{a_5}\right)^3 + f_0\frac{a_3}{a_5} - g_0\right) = 0. \tag{150}$$

$R^{(5)}$ restricted upon $\bar{c}_{\wedge^2 V}$ defines a divisor $\sim 4K_{B_2} + 2\eta$, and

$$\deg R^{(5)}|_{\bar{c}_{\wedge^2 V}} = (4K_{B_2} + 2\eta)\cdot\bar{c}_{\wedge^2 V} = D\cdot(2K_{B_2} + \eta).$$

[14] Note added in version 4: it is now understood clearly [52] why the type (a) and type (b) points do not contribute to the divisor of the line bundle (77). Ramification behavior of $C_{\wedge^2 V}$ is the key. See footnote 47 of [52].

It should be noted, however, that $R^{(5)}$ in (150) was derived under an assumption that $a_5 \neq 0$. Points on $a_5 = 0$ should not contribute to the type (c) points,[15] and one needs to examine whether $R^{(5)}|_{\bar{c}_{\wedge^2 V}}$ naively applied to the $a_5 = 0$ locus gives rise to fake contributions or not. This is carried out by taking a local coordinate on $\tilde{\bar{c}}_{\wedge^2 V}$ at around type (d) and type (a) points, respectively, and by examining whether $R^{(5)}$ has a pole or zero at these codimension-3 singularity points. After a bit of detailed analysis,[16] we see that $R^{(5)}|_{\bar{c}_{\wedge^2 V}}$ has a fake contribution of -1 at every type (a) points. Thus, the true number of type (c) points characterized by (149) and $a_5 \neq 0$ is

$$\deg R^{(5)}|_{\bar{c}_{\wedge^2 V}} + \#(a) = D \cdot (2K_{B_2} + \eta) + D \cdot \sigma,$$

which exhausts all the type (c) points expected in (148). Thus, all the type (c) points are on the $a_5 \neq 0$ part of $P^{(5)} = 0$ matter curve $\bar{c}_{\wedge^2 V}$, and are characterized by (150).[17]

To conclude, the locally free rank-1 sheaf $\widetilde{\mathcal{F}}_{\wedge^2 V}$ on $\tilde{\bar{c}}_{\wedge^2 V}$ is given by

$$\widetilde{\mathcal{F}}_{\wedge^2 V} = \mathcal{O}\left(\tilde{\imath}^*_{\wedge^2 V} K_{B_2} + \frac{1}{2}\tilde{b}^{(c)} + \tilde{\pi}_{D*}\gamma\right), \tag{151}$$

where $\tilde{\imath}_{\wedge^2 V} = i_{\wedge^2 V} \circ \nu_{\bar{c}_{\wedge^2 V}} : \tilde{\bar{c}}_{\wedge^2 V} \to \sigma$, just like in section 5.3. Table 2 shows a couple of examples of geometric data of the matter curves for different choice of the divisor η.

[15] The $a_5 = 0$ locus on the matter curve $\bar{c}_{\wedge^2 V}$ ($P^{(5)} = 0$) are classified into two groups: type (a) and type (d). The type (a) points are, by definition, different from type (c) points, and we have seen that they do not contribute to $\pi_{D*}(r|_D - R)$. Over the type (d) points, the spectral surface C_V consists of five layers of $p_{i,j,k,l,m}$ without ramification, and there is no contribution to $\pi_{D_*}(r|_D - R)$.

[16] At around a type (d) point, a_5/a_4 can be chosen as a local coordinate on each one of the two branches of $\bar{c}_{\wedge^2 V}$. Along the curve $\bar{c}_{\wedge^2 V}$, close to the type (d) point, a_3/a_5 can be treated as a finite constant value $-x_*$. Thus, $R^{(5)}/a_4^2$ neither has a pole or zero at type (d) points.
At around a type (a) point, a_5/a_3 can be chosen as a local coordinate on $\bar{c}_{\wedge^2 V}$. Because of the defining equation $P^{(5)} = a_4 a_3^2 - a_2 a_5 a_3 + a_0 a_5^2 = 0$ of the curve $\bar{c}_{\wedge^2 V}$, a_4/a_5 can be treated as a finite constant value a_2/a_3 on the curve $\bar{c}_{\wedge^2 V}$ close to the type (a) point. Thus, $R^{(5)}/a_3^2|_{\bar{c}_{\wedge^2 V}}$ has a pole of order 1 (that is, a fake contribution of -1) at every type (a) point, when it is applied naively to the type (a) points.

[17] Since $a_5 \neq 0$ and $P^5 = 0$ are assumed, the definition of $R^{(5)}$ can be modified by multiplying/dividing by a_5 or adding/subtracting by $P^{(5)}$. It is an option to take

$$R^{(5)}_{\rm mdfd} := a_5 R^{(5)} - 4\frac{a_4}{a_5}P^{(5)} = (a_2^2 - 4a_4 a_0)a_5 + (a_3^3 + f_0 a_3 a_5^2 - g_0 a_5^3).$$

$R^{(5)}_{\rm mdfd}|_{\bar{c}_{\wedge^2 V}}$ has +2 fake contribution from every type (d) point of $\bar{c}_{\wedge^2 V}$. We are benifited from (2.71) of [53], in making an improvement in version 4 here. Since the authors of [53] assigned a scaling dimension r to a_r ($r = 5, 4, 3, 0$), the first three terms have all scaling dimension 9, whereas the last two term have higher dimensions. This is why the last two terms are missing in (2.71) of [53], whereas they are retained here.

η	$\bar{c}_V \sim$	$\bar{c}_{\wedge^2 V} \sim$	#(a)	#(d)	$g(\tilde{\bar{c}}_{\wedge^2 V})$	#(c)
$10D_b + 16D_f$	D_f	$10D_b + 18D_f$	2	4	104	298
$11D_b + 15D_f$	D_b	$13D_b + 15D_f$	0	1	89	262
$11D_b + 16D_f$	$D_{b'}$	$13D_b + 18D_f$	4	7	119	334

Table 2: Examples: We chose F_1 (Hirzebruch surface) as the base manifold B_2; D_b and D_f are two independent divisors satisfying $D_b \cdot D_f = 1$, $D_b \cdot D_b = -1$ and $D_f \cdot D_f = 0$. We also use $D_{b'} \sim D_b + D_f$ in the table above. Three examples are chosen for a divisor η on B_2, so that the matter curve $\bar{c}_V \in |5K_{B_2} + \eta|$ is effective, and $|\eta|$ is base-point free, conditions derived in [22]. In all the three examples in this table, the matter curve $\bar{c}_V$ is isomorphic to $\mathbb{P}^1$ and generically smooth. On the other hand, the matter curve $\bar{c}_{\wedge^2 V}$ has # (d) > 0 double points, and is not smooth in any one of the examples. These matter curves $\bar{c}_{\wedge^2 V}$ and their normalizations, $\tilde{\bar{c}}_{\wedge^2 V}$, have very large genus.

The covering matter curve is determined through

$$\begin{aligned} 2g(\tilde{\bar{c}}_{\wedge^2 V}) - 2 &= \deg K_{\tilde{\bar{c}}_{\wedge^2 V}}, \\ &= \deg K_{\bar{c}_{\wedge^2 V}} - 2 \times \#(\mathrm{d}), && (152) \\ &= \bar{c}_{\wedge^2 V} \cdot (11K_{B_2} + 3\eta) - 2(5K_{B_2} + \eta) \cdot (3K_{B_2} + \eta), && (153) \\ &= 80K_{B_2}^2 + 47K_{B_2} \cdot \eta + 7\eta^2. && (154) \end{aligned}$$

We used the fact in the second equality that the Euler number (genus) of a curve increases by +2 (resp. -1) whenever a double point is blown up [27, 28]. On the other hand, one can calculate the following:

$$\begin{aligned} \deg \left(\tilde{\imath}^*_{\wedge^2 V} K_{B_2} + \frac{1}{2} \tilde{b}^{(c)} \right) &= \bar{c}_{\wedge^2 V} \cdot K_{B_2} + \frac{1}{2} D \cdot (\sigma + 2K_{B_2} + \eta), && (155) \\ &= 40K_{B_2}^2 + \frac{47}{2} K_{B_2} \cdot \eta + \frac{7}{2}\eta^2 = \frac{1}{2} \deg K_{\tilde{\bar{c}}_{\wedge^2 V}}. && (156) \end{aligned}$$

Thus, by applying Riemann–Roch theorem, the net chirality is given by

$$\begin{aligned} \chi(\wedge^2 V) &= \chi(\tilde{\bar{c}}_{\wedge^2 V}; \widetilde{\mathcal{F}}_{\wedge^2 V}) = [1 - g(\tilde{\bar{c}}_{\wedge^2 V})] + \deg \left(\tilde{\imath}^*_{\wedge^2 V} K_{B_2} + \frac{1}{2} \tilde{b}^{(c)} \right) + \int_{\tilde{\bar{c}}_{\wedge^2 V}} \tilde{\pi}_{D*}\gamma, \\ &= \int_{\tilde{\bar{c}}_{\wedge^2 V}} \tilde{\pi}_{D*}\gamma = D \cdot \gamma. && (157) \end{aligned}$$

It is reasonable, as we discussed right after (143), that the result is proportional to γ. See also a discussion after (172).

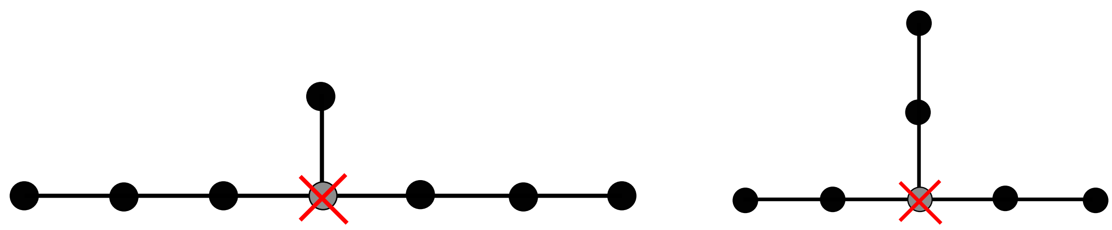


Figure 5: Extended Dynkin diagrams of (i) E_7 and (ii) E_6. By removing one node from the diagrams, one can see that these groups have subgroups $\mathrm{SU}(4) \times \mathrm{SU}(2) \times \mathrm{SU}(4) \subset E_7$, and $\mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{SU}(3) \subset E_6$.

5.5 Rank-6 Vector Bundles

The spectral surface C_V of a rank-6 bundle V is the zero locus of

$$\begin{aligned} s &= a_0 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 x^3, \\ &= \tilde{a}_0 + \tilde{a}_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2. \end{aligned} \tag{158}$$

The coefficients $a_{0,2}$ and $\tilde{a}_{0,2}$ are related through

$$a_0 = \tilde{a}_0 + a_6 g_0, \qquad a_2 = \tilde{a}_2 + a_6 f_0. \tag{159}$$

The matter curve of the fundamental representation V, $\bar{c}_V$, is given by $a_6 = 0$.

5.5.1 $\wedge^2 V$

$R^1 \pi_{Z*} \wedge^2 V$ can be studied for a rank-6 bundle, just like in the case for a rank-5 bundle. The matter curve $\bar{c}_{\wedge^2 V}$ is determined by requiring that the defining equation of the spectral surface (158) factorizes[18] locally as

$$s = (Ax + B)(Px^2 + Qy + Rx + S). \tag{160}$$

This condition is equivalent to

$$P^{(6)} := a_0 a_5^3 - a_2 a_5^2 a_3 + a_4 a_5 a_3^2 - a_6 a_3^3 = 0, \tag{161}$$

[18]The structure group of the spectral surface is reduced from $\mathrm{SU}(6)$ to $\mathrm{SU}(2) \times \mathrm{SU}(4)$. The commutant symmetry group in E_8 is enhanced from $\mathrm{SU}(2) \times \mathrm{SU}(3)$ to $\mathrm{SU}(2) \times \mathrm{SU}(4)$; note that $E_8 \supset \mathrm{SU}(2) \times E_7$, and $E_7 \supset \mathrm{SU}(4) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$, as one can see by removing one node from the extended Dynkin diagram of E_7 (see Figure 5 (i)).

and this is the defining equation of the matter curve $\bar{c}_{\wedge^2 V}$. $P^{(6)}$ is a section of $\mathcal{O}(15K_{B_2}+4\eta)$, and the curve $\bar{c}_{\wedge^2 V}$ belongs to a class $|15K_{B_2}+4\eta|$; this corresponds to the $N=6$ case of (79).

There are two different types of $\bar{c}_V$–$\bar{c}_{\wedge^2 V}$ intersection points, because $P^{(6)}|_{a_6=0}$ factorizes.

(a) $a_6 = 0$ and $a_5 = 0$,

(d) $a_6 = 0$ and $a_0 a_5^2 - a_2 a_5 a_3 + a_4 a_3^2 = P^{(5)} = 0$.

We call them type (a) and type (d) intersection points, respectively. The two curves intersect transversely at both types of intersection points. These two types exhaust all kinds of the intersection:

$$\begin{aligned}\#(\mathrm{a}) + \#(\mathrm{d}) &= (6K_{B_2}+\eta)\cdot(5K_{B_2}+\eta) + (6K_{B_2}+\eta)\cdot(10K_{B_2}+3\eta), \\ &= (6K_{B_2}+\eta)\cdot(15K_{B_2}+4\eta) = \bar{c}_V \cdot \bar{c}_{\wedge^2 V}.\end{aligned} \tag{162}$$

The matter curve $\bar{c}_{\wedge^2 V}$ itself is smooth at these intersection points.

The matter curve $\bar{c}_{\wedge^2 V}$ has a triple point, wherever

(e) $a_5 = a_3 = 0$.

The defining equation $P^{(6)}$ in (161) have three different solutions for $a_3 : a_5$ as a function of local values of $a_{0,2,4,6}$ in a local neighborhood of this type of point, and the three solutions correspond to three branches of $C_{\wedge^2 V}$ intersecting with the zero section. One corresponds to $p_i \boxplus p_j = e_0$, another to $p_k \boxplus p_l = e_0$, and the last one to $p_m \boxplus p_n = e_0$. Whenever the first two branches of the matter curve intersect in $\sigma \simeq B_2$, the last one also passes through the intersection point, because of the traceless condition (14) of rank-6 bundles. This is why $\bar{c}_{\wedge^2 V}$ for a rank-6 bundle has triple points.

In a local neighborhood of a triple point, $C_{\wedge^2 V} \subset Z$ consists of three irreducible components, one for $C_{(ij)}$, one for $C_{(kl)}$ and the other for $C_{(mn)}$. Intersection of any two of the three irreducible components are double-curve singularity of $C_{\wedge^2 V}$, and the triple points are where three double-curve singularities collide. As this type of codimension-2 singularity inevitably appears on the zero section in the case of rank-6 bundles, we need to modify the argument that we presented in section 4.1.

Only a straightforward generalization is required, however. We choose

$$\widetilde{C}_{\wedge^2 V} = C_{(ij)} \coprod C_{(kl)} \coprod C_{(mn)} \tag{163}$$

locally around any triple points. $\nu_{C_{\wedge^2 V}}$ is defined around this codimension-2 singularity by

$$\nu_{C_{ij}} \coprod \nu_{C_{kl}} \coprod \nu_{C_{mn}} : C_{(ij)} \coprod C_{(kl)} \coprod C_{(mn)} \to C_{(ij)} \cup C_{(kl)} \cup C_{(mn)} = C_{\wedge^2 V}. \tag{164}$$

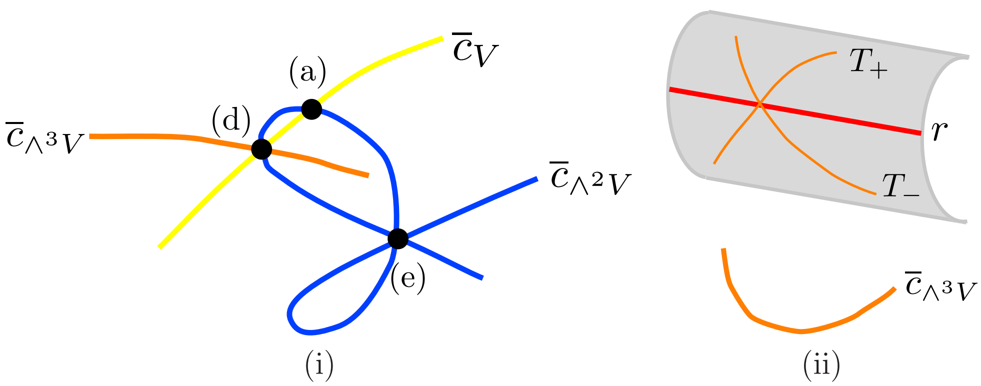


Figure 6: The left panel (i) is a schematic picture of various matter curves in the zero section for cases with rank-6 bundles. It shows how the curves intersect one another, and what kind of singularities they have. The right panel (ii) describes a geometry associated with the type (f) points, which arise in the analysis $\wedge^3 V$ bundles of rank-6 bundles V. Two irreducible components T_+ and T_- of a curve T in C_V intersect when $T_\pm$ intersects the ramification divisor r without being ramified over $\bar{c}_{\wedge^3 V}$. This is where the type (f) points are found.

By repeating almost the same argument as in section 4.1, one can see that i) $\mathcal{N}_{\wedge^2 V} = \nu_{C_{\wedge^2 V}*}\widetilde{\mathcal{N}}_{\wedge^2 V}$ exists, ii) (22) is satisfied as a sheaf of $\mathcal{O}_Z$ module, and iii) $\widetilde{\mathcal{N}}_{\wedge^2 V}$ on $\widetilde{C}_{\wedge^2 V}$ is a locally free rank-1 sheaf. Thus, (77) can be used for this case as well. The covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$ is defined as $\tilde{\bar{c}}_{\wedge^2 V} := \nu^{-1}_{C_{\wedge^2 V}}(\bar{c}_{\wedge^2 V})$ as before,[19] and each triple point is resolved into three points in $\tilde{\bar{c}}_{\wedge^2 V}$, one in $C_{(ij)}$, one in $C_{(kl)}$ and the other in $C_{(mn)}$.

The classification of the D–r intersection goes exactly the same as in the case of a rank-5 bundle V. There are type (a), (b) and (c) D–r intersection points, and only the type (c)

[19] In all the cases that we considered in this article, $\widetilde{C}_{\rho(V)}$ corresponds to normalization of $C_{\rho(V)}$. "Normalization" is a jargon in algebraic geometry which means a normal variety that is associated with an original algebraic variety. The covering matter curve $\tilde{\bar{c}}_{\rho(V)}$ is defined as the inverse image of the matter curve $\bar{c}_{\rho(V)}$ in $\nu_{C_{\rho(V)}} : \widetilde{C}_{\rho(V)} \to C_{\rho(V)}$. Reference [19] introduced a curve D'/τ as a normalization of the matter curve $\bar{c}_{\wedge^2 V}$. The two curves $\tilde{\bar{c}}_{\wedge^2 V}$ and D'/τ are the same for most of the cases, because the covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$ is obtained by resolving double points in the rank-5 case, and by resolving triple points in the rank-6 case. In the rank-4 case for $\rho(V) = \wedge^2 V$ and in the rank-6 case for $\rho(V) = \wedge^3 V$, however, these two definitions are not the same. The matter curves $\bar{c}_{\wedge^2 V}$ for the rank-4 case and $\bar{c}_{\wedge^3 V}$ for the rank-6 case are smooth, and do not need normalization.
The other definition of D'/τ in [19] can also be read out from the notation itself, a quotient of D' by τ. In all the cases we considered in this article, the covering matter curve for $\rho(V) = \wedge^2 V$ agrees with D'/τ in this definition.

η	$\bar{c}_V$	#(a)	#(d)	#(e)	$g(\tilde{\bar{c}}_{\wedge^2 V})$	#(c)	$g(\tilde{\bar{c}}_{\wedge^3 V \pm})$	#(f)
$12D_b + 19D_f$	D_f	2	16	32	261	680	270	678
$13D_b + 18D_f$	D_b	0	5	27	234	618	243	618
$13D_b + 19D_f$	$D_{b'}$	4	27	37	288	742	297	738

Table 3: Some examples of geometry that result from rank-6 bundle compactification: We chose F_1 (Hirzebruch surface) as the base manifold B_2 as before. See the caption of 2 for the definition of the divisors D_b, D_f and $D_{b'}$ of $B_2 = F_1$. Three examples are chosen for a divisor η on B_2, so that the matter curve $\bar{c}_V \in |6K_{B_2} + \eta|$ is effective, and $|\eta|$ is base-point free, conditions in [22]. Genus of the covering matter curves, numbers of various types of intersection points of those matter curves and degree of some divisors are calculated and shown for the three examples.

points contribute to $\tilde{\pi}_{D*}(r|_D - R)/2$ in (77).[20]

$$\widetilde{\mathcal{F}}_{\wedge^2 V} = \mathcal{O}_{\tilde{\bar{c}}_{\wedge^2 V}} \left(\tilde{\imath}^*_{\wedge^2 V} K_{B_2} + \frac{1}{2}\tilde{b}^{(c)} + \tilde{\pi}_{D*}\gamma|_D \right), \tag{165}$$

the same as in (151). deg $\tilde{b}^{(c)}$ is given by (148), now with $N = 6$.

[20] A footnote added in version 4: The type (c) points are characterized as follows. A pair of points p_i and p_j on C_V are on D, if and only if their coordinates on the elliptic fiber are (x_*, y_*) and $(x_*, -y_*)$, with $x_* = -a_3/a_5 = B/A$, and $y_*^2 = x_*^3 + f_0 x_* + g_0$. D is ramified over $\bar{c}_{\wedge^2 V}$ if

$$\frac{ds}{dx} = \left((a_2 + 2a_4 x_* + 3a_6 x_*^2 + a_5 y_*) + \frac{dy}{dx}(a_3 + a_5 x_*) \right) = 0.$$

The second term $(a_3 + a_5 x_*)$ vanishes, and this condition is equivalent to

$$R^{(6)} := \left(a_2 - 2a_4 \frac{a_3}{a_5} + 3a_6 \frac{a_3^2}{a_5^2} \right)^2 + a_5^2 \left(\frac{a_3^3}{a_5^3} + f_0 \frac{a_3}{a_5} - g_0 \right) = 0.$$

It should be noted that $a_5 \neq 0$ is assumed. Although $\pi_{D*}(r|_D - R)$ is not expected to leave contributions at $a_5 = 0$ locus of $\bar{c}_{\wedge^2 V}$, $R^{(6)}|_{\bar{c}_{\wedge^2 V}} = 0$ has non-zero fake contributions at type (a) points (where $a_5 = 0$), when it is applied naively to the $a_5 = 0$ locus. $R^{(5)}|_{\bar{c}_{\wedge^2 V}}$ has a pole of order 2 at every type (a) point, and the number of true type (c) points on the $a_5 \neq 0$ locus of $\bar{c}_{\wedge^2 V}$ is

$$\deg R^{(6)}|_{\bar{c}_{\wedge^2 V}} + 2\#(a) = \bar{c}_{\wedge^2 V} \cdot (4K_{B_2} + 2\eta) + 2D \cdot \sigma,$$

which is exactly the expected number of the type (c) points in (148).

The covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$ has

$$\begin{aligned} 2g(\tilde{\bar{c}}_{\wedge^2 V})-2 &= \deg K_{\tilde{\bar{c}}_{\wedge^2 V}}, \\ &= \deg K_{\bar{c}_{\wedge^2 V}} - 6\times\#(\mathrm{e}), && (166)\\ &= (15K_{B_2}+4\eta)\cdot(16K_{B_2}+4\eta)-6(5K_{B_2}+\eta)\cdot(3K_{B_2}+\eta), && (167)\\ &= 150K_{B_2}^2+76K_{B_2}\cdot\eta+10\eta^2. && (168)\end{aligned}$$

In the second equality, we have used the fact that the genus of a curve reduces by 3 when a triple point is blown up; see [27, 28]. On the other hand,

$$\begin{aligned} \deg\left(\tilde{\imath}^*_{\wedge^2 V}K_{B_2}+\frac{1}{2}\tilde{b}^{(c)}\right) &= (15K_{B_2}+4\eta)\cdot K_{B_2}+\frac{1}{2}D\cdot(2\sigma+2K_{B_2}+\eta), && (169)\\ &= 75K_{B_2}^2+38K_{B_2}\cdot\eta+5\eta^2=\frac{1}{2}K_{\tilde{\bar{c}}_{\wedge^2 V}}. && (170)\end{aligned}$$

Thus, using the Riemann–Roch theorem, the net chirality from the bundle $\wedge^2 V$ is given by

$$\chi(\wedge^2 V)=\chi(\tilde{\bar{c}}_{\wedge^2 V};\widetilde{\mathcal{F}}_{\wedge^2 V})=\int_{\tilde{\bar{c}}_{\wedge^2 V}}\tilde{\pi}_{D*}\gamma=D\cdot\gamma. \qquad (171)$$

After studying the direct images $R^1\pi_{Z*}\wedge^2 V$ one by one for V of various ranks, we find the the net chirality from these bundles is given by the same expression, $\chi(\wedge^2 V)=D\cdot\gamma$. It will be clear that the rank-4 (143), rank-5 (157) and rank-6 (171) cases have this form of expression. In the rank-3 case, $\chi(\wedge^2 V)=\int_{\bar{c}_{\wedge^2 V}}-j^*\gamma+\pi_{C*\gamma}=D\cdot\gamma$, too. Thus, it is tempting to guess that

$$\chi(\wedge^2 V)=D\cdot\gamma \qquad (172)$$

for any $\mathrm{U}(N)$ bundles given by spectral cover construction.

In order to show that this is really the case for general N, we need a better way to obtain $\deg K_{\tilde{\bar{c}}_{\wedge^2 V}}$. The relation between $\deg K_{\bar{c}_{\wedge^2 V}}$ and $\deg K_{\tilde{\bar{c}}_{\wedge^2 V}}$ was different for all the different rank V we have considered. To avoid this rank V dependence, the following observation in [19] is useful: $\tilde{\pi}_D : D\to\tilde{\bar{c}}_{\wedge^2 V}$ is a degree-2 cover for any rank $V=N$, and

$$\begin{aligned} \deg K_{\tilde{\bar{c}}_{\wedge^2 V}} &= \frac{1}{2}\left(\deg K_D-\deg R\right), && (173)\\ &= \frac{1}{2}\left(D\cdot(2C_V-(\sigma+\sigma'))-\deg R\right). && (174)\end{aligned}$$

Using a relation $R=(\sigma+\sigma')|_D$ [18], this expression can be also rewritten as

$$\deg K_{\tilde{\bar{c}}_{\wedge^2 V}}=D\cdot C_V-\deg R. \qquad (175)$$

On the other hand,

$$\deg\left(\tilde{\imath}^*_{\wedge^2 V} K_{B_2} + \frac{1}{2}\tilde{\pi}_{D*}\left(r|_D - R\right)\right) = \frac{1}{2}\left(D \cdot (\pi_D^* K_{B_2} + r) - \deg R\right) = \frac{1}{2}\left(D \cdot C_V - \deg R\right). \tag{176}$$

Thus, we always have

$$\deg\left(\tilde{\imath}^*_{\wedge^2 V} K_{B_2} + \frac{1}{2}\tilde{\pi}_{D*}(r|_D - R)\right) = \frac{1}{2}\deg K_{\tilde{\bar{c}}_{\wedge^2 V}}. \tag{177}$$

Now, the chirality formula (172) follows from (77) and the Riemann–Roch theorem for general rank $V = N$.

The net chirality in the matter multiplets from bundle V is given by (36), and that in the matter multiplets from bundle $\wedge^2 V$ by (172). Although both are determined by one and the same γ, it is not obvious what kind of relations the net chiralities in the two sectors satisfy. We know, if we calculate $\chi(Z;V)$ and $\chi(Z;\wedge^2 V)$ by applying the Hirzebruch–Riemann–Roch theorem on a Calabi–Yau 3-fold, that a relation (375) should hold between them. Thus, we will show how the relation (375) follows also from the chirality formulae (36) and (172).

First, note that γ can be decomposed into

$$\gamma = \gamma_0 + \pi_Z^*\omega, \qquad \gamma_0 = \lambda(N\sigma - \eta + NK_{B_2}) \tag{178}$$

for some λ and a 2-form ω on B_2. Since $\pi_{C*}\gamma_0 = 0$ and $\pi_{C*}\pi_Z^*\omega = N\omega$, only the$\gamma_0$ part is allowed for SU(N) bundles [12]. For SU(N) bundles,

$$\chi(V) = \bar{c}_V \cdot \gamma_0 = -\lambda\eta \cdot (NK_{B_2} + \eta) \tag{179}$$

from (36), and

$$\begin{aligned}
\chi(\wedge^2 V) &= D \cdot \gamma_0 \\
&= [\sigma \cdot (N(N-1)K_{B_2} + 2(N-2)\eta) + \eta \cdot (3K_{B_2} + \eta)] \cdot \lambda(N\sigma - \eta - NK_{B_2}), \\
&= \lambda\left(-\eta \cdot (N(N-1)K_{B_2} + 2(N-2)\eta) + N\eta \cdot (3K_{B_2} + \eta)\right), && (180) \\
&= -\lambda\eta \cdot (NK_{B_2} + \eta) \times (N-4) && (181)
\end{aligned}$$

from (172). Thus, the expression (172) yields a result consistent with the relation $\chi(\wedge^2 V) = (N-4)\chi(V)$ in (375) for the case with $c_1(V) = 0$. It is also easy to show (through a similar calculation) that

$$D \cdot \pi_Z^*\omega = (N-4) \times \bar{c}_V \cdot \omega + (3K_{B_2} + \eta) \cdot (N\omega). \tag{182}$$

Therefore, the chirality formula (172) for U(N) bundles always yields a result consistent with (375).

5.5.2 $\wedge^3 V$

In the Heterotic string compactification with an SU(6) bundle V in E_8, another species of chiral multiplets arise from the cohomology group $H^1(Z; \wedge^3 V)$. Thus, the direct image $R^1\pi_{Z*} \wedge^3 V$ is studied here, so that its F-theory description is obtained.

The matter curve $\bar{c}_{\wedge^3 V}$ is characterized by a condition that the defining equation of the spectral surface (158) factorizes[21] locally as

$$s = (Ay + Bx + C)(Py + Qx + R). \tag{183}$$

Three points p_i, p_j, p_k satisfying $(Ay + Bx + C) = 0$ satisfy $p_i \boxplus p_j \boxplus p_k = e_0$. After some calculations, one finds that this factorization condition is equivalent to

$$Q^{(6)} := a_6(\tilde{a}_2^2 - 4a_4\tilde{a}_0) + (\tilde{a}_0 a_5^2 - \tilde{a}_2 a_5 a_3 + a_4 a_3^2) = 0. \tag{184}$$

$Q^{(6)}$ is a section of $\mathcal{O}(10K_{B_2} + 3\eta)$, and $\bar{c}_{\wedge^3 V}$ belongs to a class $|10K_{B_2} + 3\eta|$. This curve passes through the type (d) $\bar{c}_V$–$\bar{c}_{\wedge^2 V}$ intersection points, and it intersects with $\bar{c}_V$ only at such points. This is because $Q^{(6)}|_{a_6=0} = P^{(5)}$. See Figure 6.

The prescription of [18] in determining the topological class of $C_{\rho(V)}$ is that

$$C_{\rho(V)} \sim (\mathrm{dim}.\rho(V))\sigma + 2T_{\rho(V)}\pi_Z^*\eta, \tag{185}$$

where $T_{\rho(V)}$ is the Dynkin index of representation ρ. Because

$$\mathrm{dim}.\wedge^3 V = \frac{N(N-1)(N-2)}{3!}, \qquad T_{\wedge^3 V} = \frac{(N-2)(N-3)}{4} \tag{186}$$

for $\wedge^3 V$ of a rank-N bundle V, the naive expectation is that $\bar{c}_{\wedge^3 V} \sim C_{\wedge^3 V} \cdot \sigma \sim (20K_{B_2} + 6\eta)$. This is twice as much as the result that we have obtained. This is because $\bar{c}_{\wedge^3 V}$ is actually a double curve in $C_{\wedge^3 V}$, just like $\bar{c}_{\wedge^2 V}$ is in $C_{\wedge^2 V}$ of a rank-4 bundle V. The three points $\{p_l, p_m, p_n\}$ satisfying $(Py + Qx + R) = 0$ also satisfy $p_l \boxplus p_m \boxplus p_n = e_0$ simultaneously.

An idea was presented in section 4 how to study $R^1\pi_{Z*}\wedge^2 V$. The same idea can be applied to $R^1\pi_{Z*}\wedge^3 V$ only with quite a natural generalization. The treatment in section 4 allows us to obtain a locally free rank-1 sheaf $\widetilde{\mathcal{F}}_{\wedge^3 V}$ on a covering matter curve $\tilde{\bar{c}}_{\wedge^3 V}$, if the two conditions are satisfied: i) the Fourier–Mukai transform of $\wedge^3 V$ on Z is represented as a pushforward (as

[21] The structure group of the spectral surface reduces from SU(6) to SU(3) × SU(3), and the commutant enhanced from SU(3) × SU(2) to SU(3) × SU(3); note that $E_8 \supset \mathrm{SU}(3) \times E_6$, and $E_6 \supset \mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{SU}(3)$, as one can see by removing one node from the extended Dynkin diagram of E_6.

in (22)) as a sheaf of $\mathcal{O}_Z$-module, and ii) $\mathcal{N}_{\wedge^3 V}$ on $C_{\wedge^3 V}$ is given by a pushforward of a locally free rank-1 sheaf $\widetilde{\mathcal{N}}_{\wedge^3 V}$ on $\widetilde{C}_{\wedge^3 V}$, a resolution of $C_{\wedge^3 V}$. In the situation we have, the matter curve $\bar{c}_{\wedge^3 V}$ itself is a double curve in $C_{\wedge^3 V}$, but this double-curve singularity is resolved by blowing up Z with a center along the double-curve singularity, just like in the appendix B.3 where $\wedge^2 V$ bundle for a rank-4 bundle V was discussed. We now have a covering curve $\tilde{\bar{c}}_{\wedge^2 V}$, which is a degree-2 cover of $\bar{c}_{\wedge^2 V}$. Furthermore, since $[A : B : C] = [P : Q : R]$ can be realized only on a codimension-2 locus in curve $\bar{c}_{\wedge^3 V}$, the degree-2 cover does not ramify for a generic choice of moduli parameters $a_{0,2,3,4,5,6}$. The covering matter curve is a disjoint union of two copies of $\bar{c}_{\wedge^3 V}$:

$$\tilde{\bar{c}}_{\wedge^3 V} = \tilde{\bar{c}}_{\wedge^3 V+} \coprod \tilde{\bar{c}}_{\wedge^3 V-}. \tag{187}$$

We have no reason to expect that singularities appear on these curves. Therefore, no extra complication arises other than the original double-curve singularity, and we have shown in section 4 how to deal with double-curve singularity; thus, the idea in section 4 is now applicable to the analysis of $R^1\pi_{Z*} \wedge^3 V$ for a rank-6 bundle V.

Instead of a curve D in $Y = \pi_Z^{-1}(\bar{c}_{\wedge^2 V})$, a curve T in $Y = \pi_Z^{-1}(\bar{c}_{\wedge^3 V})$ is introduced. A triplet of points $\{p, p', p''\}$ in $C_V|_{E_b}$ ($b \in \bar{c}_{\wedge^3 V}$) satisfying $p \boxplus p' \boxplus p'' = e_0$ sweeps a curve in Y, and that is the definition of T. $\pi_T = \pi_Z|_T : T \to \bar{c}_{\wedge^3 V}$ is not necessarily a degree-3 cover, but a projection to the covering curve $\tilde{\pi}_{\tilde{T}} : \tilde{T} \to \tilde{\bar{c}}_{\wedge^3 V}$ is a degree-3 cover. $\tilde{T}$ is a resolution of T as we will explain it later. For the case of a rank-6 bundle V, the three solutions of $(Ay + Bx + C) = 0$ [resp. of $(Py + Qx + R) = 0$] form T_+ part [resp. T_- part] of $T = T_+ \cup T_-$, and $\tilde{T} = T_+ \coprod T_-$. $T_\pm$ is mapped to $\tilde{\bar{c}}_{\wedge^3 V\pm}$ separately.

A locally free rank-1 sheaf $\widetilde{\mathcal{F}}_{\wedge^3 V}$ on $\tilde{\bar{c}}_{\wedge^3 V}$ is given by

$$\widetilde{\mathcal{F}}_{\wedge^3 V} = \mathcal{O}\left(\tilde{\imath}^*_{\wedge^3 V} K_{B_2} + \tilde{\pi}_{\tilde{T}*}\left(\frac{1}{2}(r|_{\tilde{T}} - R_{(T)}) + \gamma|_T\right)\right), \tag{188}$$

a straightforward generalization of the discussion that has led to (77). A divisor $R_{(T)}$ is a ramification divisor of $\tilde{\pi}_{\tilde{T}} : \tilde{T} \to \tilde{\bar{c}}_{\wedge^3 V}$, and hence $R_{(T)} := K_{\tilde{T}} - \tilde{\pi}^*_{\tilde{T}} K_{\tilde{\bar{c}}_{\wedge^3 V}}$.

For the rank-6 case, the covering matter curve is a disjoint union of two curves, $\tilde{\bar{c}}_{\wedge^3 V\pm}$, and each curve has a locally free rank-1 sheaf

$$\widetilde{\mathcal{F}}_{\wedge^3 V\pm} = \mathcal{O}\left(\tilde{\imath}^*_{\wedge^3 V\pm} K_{B_2} + \tilde{\pi}_{T\pm*}\left(\frac{1}{2}(r|_{T_\pm} - R_{(T_\pm)}) + \gamma|_{T_\pm}\right)\right), \tag{189}$$

where $\tilde{\pi}_{T\pm} := \tilde{\pi}_{\tilde{T}}|_{T_\pm}$ maps $T_\pm$ to $\tilde{\bar{c}}_{\wedge^3 V\pm}$, $R_{(T_\pm)}$ their ramification divisors, and $r|_{T_\pm}$ a restriction on $T_\pm$ of a pullback of $r|_T$ to $\tilde{T}$. $\tilde{\imath}^*_{\wedge^3 V\pm}$ denotes pullback via either one of $\tilde{\imath}_{\wedge^3 V\pm} := (i_{\wedge^3 V} \circ \nu_{\bar{c}_{\wedge^3 V}})|_{\tilde{\bar{c}}_{\wedge^2 V}}$.

The remaining task is to understand the divisor $\tilde{\pi}_{T_\pm *}(r|_{T_\pm} - R_{(T_\pm)})$ better. Let us begin with examining $R_{(T_\pm)}$. We first count the number of points where the projection $\tilde{\pi}_{\tilde{T}}$ is ramified. From the definition of the ramification divisor, we have

$$\deg R_{(T)} = \deg K_{\tilde{T}} - 6 \times \deg K_{\bar{c}_{\wedge^3 V}}; \tag{190}$$

the second term needs a factor 6 because $\tilde{\pi}_{\tilde{T}}$ is a degree-3 cover of $\tilde{\bar{c}}_{\wedge^3 V}$, and the latter consists of two copies of $\bar{c}_{\wedge^3 V}$. The second term is easy to calculate:

$$\deg K_{\bar{c}_{\wedge^3 V \pm}} = \deg K_{\bar{c}_{\wedge^3 V}} = (10K_{B_2} + 3\eta) \cdot (11K_{B_2} + 3\eta). \tag{191}$$

Next, let us calculate $\deg K_T$. Since the curve T collects all the points in $C_V \cdot Y$ (for a rank-6 bundle V), T is topologically

$$T \sim C_V \cdot \pi_Z^{-1}(10K_{B_2} + 3\eta) = (6\sigma + \eta) \cdot (10K_{B_2} + 3\eta). \tag{192}$$

Applying the adjunction formula to the curve T in C_V (or in Y), we have

$$\deg K_T = (10K_{B_3} + 3\eta) \cdot (6\sigma + \eta) \cdot (6\sigma + \eta + 10K_{B_2}) = (10K_{B_2} + 3\eta) \cdot 6(16K_{B_2} + 5\eta). \tag{193}$$

The curve T has two irreducible components, T_+ and T_-, and the two components intersect at some points. $\tilde{T}$ is obtained by resolving the double points formed by T_+ and T_-. As we will see later,

$$T_+ \cdot T_- = (10K_{B_2} + 3\eta) \cdot (9K_{B_2} + 3\eta). \tag{194}$$

The genus (resp. $\deg K$) of a curve decreases by 1 (resp. 2) when a double point is blown up. Thus,

$$\deg K_{\tilde{T}} = \deg K_T - 2T_+ \cdot T_- = (10K_{B_2} + 3\eta) \cdot (78K_{B_2} + 24\eta). \tag{195}$$

Combining all the information we have obtained, one finds that

$$\deg R_{(T)} = (10K_{B_2} + 3\eta) \cdot (12K_{B_2} + 6\eta). \tag{196}$$

$\deg R_{(T)}$ above contains both $\deg R_{(T_+)}$ and $\deg R_{(T_-)}$:

$$\begin{aligned} \deg R_{(T)} &= \deg R_{(T_+)} + \deg R_{(T_-)}, \\ &= (10K_{B_2} + 3\eta) \cdot (6K_{B_2} + 6\eta_+) + (10K_{B_2} + 3\eta) \cdot (6K_{B_2} + 6\eta_-), \end{aligned} \tag{197}$$

where $\eta_+ + \eta_- = \eta$. Although we are not presenting details, these intersection numbers can be understood as the number of points in $\bar{c}_{\wedge^3 V \pm}$ where $(Ay + Bx + C) = 0$ [resp.

$(Py+Qx+R)=0]$ has a double root in their fiber, and hence T_+ (resp. T_-) ramifies. Since a_0 is a global section of $\mathcal{O}(\eta)$, C and R are sections (only along $\bar{c}_{\wedge^3 V}$) of $\mathcal{O}(\eta_+)$ and $\mathcal{O}(\eta_-)$, respectively. Divisors $\eta_\pm$ are defined on $\bar{c}_{\wedge^3 V}$, and $\eta_+ + \eta_- = \eta|_{\bar{c}_{\wedge^3 V}}$.

Whenever curves $T_\pm$ in C_V ramify over $\tilde{\bar{c}}_{\wedge^3 V\pm}$, the spectral surface C_V does the same over the zero section σ. Thus, such ramification points also contribute to deg $r|_T$. The entire contribution to deg $r|_T$ is given by

$$\begin{aligned} \deg r|_T &= T\cdot r = (6\sigma+\eta)\cdot(10K_{B_2}+3\eta)\cdot(6\sigma+\eta-K_{B_2}), \\ &= (10K_{B_2}+3\eta)\cdot(30K_{B_2}+12\eta). \end{aligned} \tag{198}$$

Assuming that T–r intersection takes place with multiplicity 1 at all the ramification points of $T_\pm$, we find that

$$\deg r|_T - \deg R_{(T)} = (10K_{B_2}+3\eta)\cdot(18K_{B_2}+6\eta) \tag{199}$$

remains. This should be the contributions to deg $r|_T$ that are not from the ramification points of $T_\pm$.

When T_+ intersects with the ramification divisor r on C_V at a point where $\tilde{\pi}_+ : T_+ \to \tilde{\bar{c}}_{\wedge^3 V+}$ is not ramified, T_- also runs through the same point; see Figure 6. Such points contribute to deg $r|_T$ = deg $r|_{T_\pm}$, but not to deg $R_{(T_\pm)}$. Let us find out where in $\bar{c}_{\wedge^3 V}$ we should expect this to happen, and how many such points there are. At such a point, $(Ay+Bx+C)=0$ and $(Py+Qx+R)=0$ share a same root, (x_*, y_*). Thus,

$$\begin{pmatrix} C \\ R \end{pmatrix} = -\begin{pmatrix} A & B \\ P & Q \end{pmatrix}\begin{pmatrix} y_* \\ x_* \end{pmatrix}, \quad \text{and hence} \quad \begin{pmatrix} y_* \\ x_* \end{pmatrix} = \frac{-1}{AQ-BP}\begin{pmatrix} Q & -B \\ -P & A \end{pmatrix}\begin{pmatrix} C \\ R \end{pmatrix}. \tag{200}$$

Since (x_*, y_*) expressed in terms of A, B, C, P, Q, R should satisfy the Weierstrass equation of the elliptic curve, we have an equation constraining $A \sim R$. Writing the equation explicitly,

$$S^{(6)} := -(AQ-BP)(BR-CQ)^2+(CP-AR)^3+f_0(AQ-BP)^2(CP-AR)+g_0(AQ-BP)^3 = 0. \tag{201}$$

$S^{(6)}$ is a section of $\mathcal{O}(9K_{B_2}+3\eta)|_{\bar{c}_{\wedge^3 V}}$. Let us denote the divisor of the zero locus of $S^{(6)}$ as $b^{(f)}$. Therefore,

$$\deg b^{(f)} = (10K_{B_2}+3\eta)\cdot(9K_{B_2}+3\eta). \tag{202}$$

For each zero locus of $S^{(6)}$, T_+, T_- and r intersect in C_V (as in Figure 6), and all these type (f) intersection points give rise to the contributions (194) and (199). All the contributions to

$\deg r|_T$ and $\deg R_{(T)}$ are now understood, and we conclude that

$$\tilde{\pi}_{T_\pm *}(r|_{T_\pm} - R|_{(T_\pm)}) = \tilde{b}^{(f)}_\pm, \tag{203}$$

where $\tilde{b}^{(f)}_\pm := (\nu_{\bar{c}_{\wedge^3 V}}|_{\tilde{\bar{c}}_{\wedge^3 V\pm}})^{-1}(b^{(f)})$.

Therefore, we are now ready to write down the line bundles on the covering matter curves:

$$\widetilde{\mathcal{F}}_{\wedge^3 V\pm} = \mathcal{O}\left(\tilde{i}^*_{\wedge^2 V\pm} K_{B_2} + \frac{1}{2}\tilde{b}^{(f)}_\pm + \tilde{\pi}_{T_\pm *}\gamma|_{T_\pm}\right). \tag{204}$$

$\mathcal{F}_{\wedge^3 V}$ on the matter curve $\bar{c}_{\wedge^3 V}$ is given by a pushforward of the two line bundles $\widetilde{\mathcal{F}}_{\wedge^3 V\pm}$, and hence becomes a direct product of two line bundles. Massless chiral multiplets are identified with

$$H^1(Z; \wedge^3 V) \cong H^0(\tilde{\bar{c}}_{\wedge^3 V+}; \widetilde{\mathcal{F}}_{\wedge^3 V+}) \oplus H^0(\tilde{\bar{c}}_{\wedge^3 V-}; \widetilde{\mathcal{F}}_{\wedge^3 V-}). \tag{205}$$

It is now straightforward to see that

$$\deg\left(i^* K_{B_2} + \frac{1}{2}\tilde{b}^{(f)}_\pm\right) = (10K_{B_2} + 3\eta)\cdot\frac{1}{2}(2K_{B_2} + (9K_{B_2} + 3\eta)) = \frac{1}{2}\deg K_{\tilde{\bar{c}}_{\wedge^3 V\pm}}. \tag{206}$$

Therefore,

$$\begin{aligned}
\chi(\wedge^3 V) &= \chi(\tilde{\bar{c}}_{\wedge^3 V+}; \widetilde{\mathcal{F}}_{\wedge^3 V+}) + \chi(\tilde{\bar{c}}_{\wedge^3 V-}; \widetilde{\mathcal{F}}_{\wedge^3 V-}), \\
&= \int_{\tilde{\bar{c}}_{\wedge^3 V+}} \tilde{\pi}_{T_+ *}\gamma + \int_{\tilde{\bar{c}}_{\wedge^3 V-}} \tilde{\pi}_{T_- *}\gamma = T_+\cdot\gamma + T_-\cdot\gamma = \int_{\bar{c}_{\wedge^3 V}} \pi_{T*}\gamma.
\end{aligned} \tag{207}$$

For physics application that we mentioned in section 2 (in Table 1), $\wedge^3 V$ bundle of a rank-6 bundle is purely of SU(6) bundle V; even when a structure group of V is chosen to be $\mathrm{U}(6) \subset \mathrm{SO}(12)$, the bundle $\wedge^3 V$ is neutral under the U(1) symmetry in the structure group. Thus, $\pi_{C*}\gamma = 0$ should be use for the calculation of chirality here, and hence $\chi(\wedge^3 V) = 0$. This should be the case, since coming out of the bundle $\wedge^3 V$ are chiral multiplets in the doublet representation of an unbroken SU(2) gauge group, and there is no well-defined chirality associate with this representation (or gauge group). This serves as a consistency check, giving a confidence in the description of the bundles we have provided.

6 From Heterotic String to F-theory

The Heterotic string theory compactified on an elliptic fibered manifold Z has a dual description in F-theory. The matter curves $\bar{c}_{\rho(V)}$, the support of $R^1\pi_{Z*}\rho(V)$ in the Heterotic

theory description, correspond to intersection curves of 7-branes in F-theory. Sheaves on the matter curves, $\mathcal{F}_{\rho(V)}$, obtained in the Heterotic theory are also believed to be shared by the dual F-theory description.

In the previous section, a detailed description of $\mathcal{F}_{\rho(V)}$ was obtained in terms of spectral surface C_V and γ. The geometric data, C_V and γ, were introduced to describe vector bundles on Z, and hence the description of $\mathcal{F}_{\rho(V)}$ in the previous section is still phrased in terms of data of Heterotic string compactification. We will take necessary steps in this section to translate the description of $\mathcal{F}_{\rho(V)}$ into F-theory language.

A dictionary for the translation already exists since 1990's. The holomorphic sections a_r ($r = 0, 2, 3, \cdots, N$) become a part of complex structure moduli of an elliptic Calabi–Yau 4-fold for the dual F-theory compactification [4, 6, 7, 12, 14, 15], and γ corresponds to four-form flux G in F-theory [15]. We find, however, that the dictionary has to be refined in order to complete the translation, and that is what we do in sections 6.1 and 6.2. After the dictionary is completed, we will see in section 6.3 that some components of the divisors describing $\mathcal{F}_{\rho(V)}$ correspond to codimension-3 singularities in F-theory geometry X.

6.1 Describing Vector Bundles via dP_8 Fibration

Reference [12] explains how a del Pezzo surface dP_r describes flat bundles on an elliptic curve, and Ref. [15, 10] refined the correspondence between the moduli space of complex structure of dP_8 and data determining spectral surface of SU(N) bundles, but details are left to readers. In the first subsection of section 6, we begin with filling the details that were not spelt out explicitly in the literature, so that ordinary physicists (like majority of the authors of this article) can understand.

We denote a del Pezzo surface dP_8 as S. Its second cohomology group is generated by L_0 and L_I ($I = 1, 2, \cdots, 8$), with their intersection form given by

$$L_0 \cdot L_0 = 1, \qquad L_0 \cdot L_I = 0, \qquad L_I \cdot L_J = -\delta_{IJ} \qquad (\text{for } 1 \leq I, J \leq 8). \tag{208}$$

The anti-canonical divisor of S is given by

$$x := -K_S = 3L_0 - \sum_{I=1}^{8} L_I. \tag{209}$$

General elements E of the class $|x|$ is a curve of genus 1.

The subsets of $H^2(S;\mathbb{Z})$, I_8 and R_8, are defined as follows:

$$I_8 := \left\{ l \in H^2(S;\mathbb{Z}) | l \cdot l = -1, \quad l \cdot x = 1 \right\}, \tag{210}$$
$$R_8 := \left\{ C \in H^2(S;\mathbb{Z}) | C \cdot C = -2, \quad C \cdot x = 0 \right\}. \tag{211}$$

Elements of I_8 and R_8 are in one to one correspondence through $l = C + x$. $R_8 \otimes \mathbb{Z}$ is the subspace of $H^2(S;\mathbb{Z})$ orthogonal to x in the intersection form, and it is known that the intersection form restricted on $H^2(S;\mathbb{Z})^\perp$ is given by the Cartan matrix of E_8 Lie algebra multiplied by (-1). Elements of R_8 (and hence those of I_8) are in one to one correspondence with roots of E_8 Lie algebra.

$$C_I = L_I - L_{I+1} \quad (\text{for } I = 1, \cdots, 7) \qquad \text{and} \qquad C_8 = L_0 - (L_1 + L_2 + L_3) \tag{212}$$

can be chosen as the generators of R_8 (and of the root lattice).[22] When a complex structure of S is given (with an elliptic curve $E \in |x|$ embedded in S), a flat bundle on E is given by [12]

$$\mathcal{O}(\text{div } C_\alpha|_E) \simeq \mathcal{O}(p_\alpha - e_0) \tag{213}$$

for a root α of $\mathfrak{e}_8$; here, C_α is an element of R_8 that corresponds to α. p_α is a point on E given by $l_\alpha \cdot E$, and e_0 is the unique base point of $|x|$.

Spectral surface describes a bundle on an elliptic fibration $\pi_Z : Z \to B_2$ by specifying a set $\{p_\alpha\}_{\alpha \in R_8}$ for each E_b ($b \in B_2$). The same role can be played by a dP_8 fibration $\pi_U : U \to B_2$. Z is identified with a subset of U, so that $\pi_U|_Z = \pi_Z$. Elliptic fiber $E_b = \pi_Z^{-1}(b)$ is a subset of a dP_8 surface $S_b := \pi_U^{-1}(b)$, and complex structure of S_b determines a flat bundle on E_b through (213).

6.1.1 Two Descriptions of a dP_8 Surface

A dP_8 surface S can be described in two different ways, each of which has its own advantage.

A: S is given by blowing up 8 points $p_{1,2,\cdots,8}$ in $\mathbb{P}^2$; $\pi : S \to \mathbb{P}^2$,

B: S is a subvariety of $W\mathbb{P}^3_{1,1,2,3}$ given by an equation of homogeneous degree 6.

The description A is useful in capturing the 240 (-1)-lines of I_8, while it is easier to identify the elliptic curve E in the description B.

The two descriptions are related as follows. In the description A, L_0 is a line of $\mathbb{P}^2$, and L_I ($I = 1, \cdots, 8$) are exceptional curves, $L_I = \pi^{-1}(p_I)$ (set theoretic inverse image). There

[22] Root lattices of E_r ($r = 6, 7, 8$) are generated by C_I ($I = 1, \cdots, (r-1)$) and C_8.

are two independent global holomorphic sections in $H^0(S;\mathcal{O}(x))$; there are ${}_5C_2 - 8 = 2$ degrees of freedom in a cubic form on $\mathbb{P}^2$ that have all p_I's as zeros of order one. Let us take their generators as F_0 and F_1. Similarly, there are ${}_8C_2 - 8 \times 3 = 4$ degrees of freedom in $H^0(S;\mathcal{O}(2x))$, and we choose a generator G so that $H^0(S;\mathcal{O}(2x))$ is generated by G, F_0^2, F_0F_1 and F_1^2. Through a similar argument, one also finds that another generator—which we denote as H—is necessary for $H^0(S;\mathcal{O}(3x))$. A map $\Phi : S \to W\mathbb{P}^3_{1,1,2,3}$ is given by

$$\Phi : S \ni s \mapsto [Z' : Z : X : Y] = [F_0(s) : F_1(s) : G(s) : H(s)] \in W\mathbb{P}^3_{1,1,2,3}, \tag{214}$$

where $[Z' : Z : X : Y]$ are homogeneous coordinates of $W\mathbb{P}^3_{1,1,2,3}$. $\Phi(S)$ does not occupy the entire weighted projective space $W\mathbb{P}^3_{1,1,2,3}$. Because

$$\dim_{\mathbb{C}} H^0(S;\mathcal{O}(6x)) = {}_{20}C_2 - 8 \times \frac{6 \times 7}{2} = 22, \tag{215}$$

there must be an algebraic relation among 23 monomials of homogeneous degree 6 made out of F_0, F_1, G and H:

$$H^2 + (c_0F_0 + c_1F_1)HG + \cdots = 0. \tag{216}$$

Thus, the image $\Phi(S)$ is mapped in a subspace of $W\mathbb{P}^3_{1,1,2,3}$ given by an equation obtained by replacing F_0, F_1, G and H in the relation above by homogeneous coordinates Z', Z, X and Y:

$$Y^2 + (c_0Z' + c_1Z)YX + \cdots = 0. \tag{217}$$

Complex structure moduli of a dP_8 are described by eight complex parameters. In the description A, 16 complex numbers are needed to specify 8 points in $\mathbb{P}^2$, but there is redundancy of $PGL_3\mathbb{C}$, which is of dimension 8. Thus, the dimension of the moduli space is 8. One arrives at the same conclusion in the description B. The defining equation of a dP_8 surface in $W\mathbb{P}^3_{1,1,2,3}$ can be cast into Weierstrass form

$$Y^2 = X^3 + F^{(4)}(Z', Z)X + G^{(6)}(Z', Z) \tag{218}$$

by redefining Y and X. $F^{(4)}$ and $G^{(6)}$ are homogeneous function of Z and Z' and are of degree 4 and 6, respectively. Thus, they are described by $5 + 7 = 12$ complex numbers. Since the $GL_2\mathbb{C}$ coordinate transformation of (Z, Z') can be still used to reduce the freedom, there are 8 moduli parameters left. Those eight moduli correspond to those of E_8 flat bundles on an elliptic curve E.

6.1.2 SU(5) Bundle in Description A

For physics application, it is often more interesting to think of a bundle with smaller structure group, because the commutant of the structure group is left unbroken and can be seen in low-energy physics. We are definitely interested in such situations for phenomenological applications. Smaller structure group corresponds to a restricted moduli space. We take $\mathrm{SU}(5)_{\mathrm{bdl}}$ ($\subset E_8$) structure group as an example; this is certainly the most motivated case in phenomenology.

In the description A, the $\mathrm{SU}(5)_{\mathrm{bdl}}$ structure group of a flat bundle on E corresponds to choosing four points p_{A+1} ($A = 1, \cdots, 4$) infinitesimally near p_A, i.e., on $\mathbb{P}^1$ that is obtained by blowing up p_A. Because these four points are chosen within $\mathbb{P}^1$'s and not from the entire $\mathbb{P}^2$, the dimension of the moduli space is reduced by four, leaving $8 - 4 = 4$ moduli. This agrees with our expectation coming from $\dim_{\mathbb{C}} \mathbb{P}^4 = 4$, the dimension of the moduli space of flat $\mathrm{SU}(5)_{\mathrm{bdl}}$ bundles on E. Among the generators of R_8, $C_A = L_A - L_{A+1}$ for $A = 1, 2, 3, 4$ are $\mathbb{P}^1$ obtained right after blowing up the point p_A, and are effective curves. Their intersection form is the $(-1)\times$ Cartan matrix of $\mathfrak{su}(5)_{\mathrm{GUT}}$. When the $\mathbb{P}^1$ cycles obtained by the first four blow-ups are of zero size, then S develops an A_4 type singularity. dP_8 surface contains an element of I_8

$$l_0 := 3L_0 - 2L_1 - L_2 - L_3 - L_4 - L_6 - L_7 - L_8, \tag{219}$$

which is[23] a cubic curve in $\mathbb{P}^2$ that has a double point at p_1. Intersection diagram of l_0 (or $C_0 := l_0 - x = -L_1 + L_5$) and C_A ($A = 1, 2, 3, 4$) forms the extended Dynkin diagram of A_4. Since

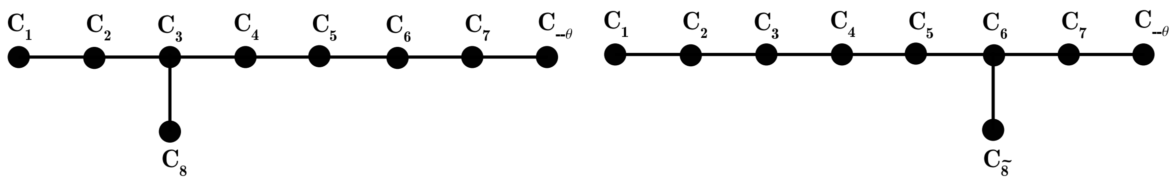
$$l_0 + \sum_{A=1}^{4} C_A = x, \tag{220}$$

one of generators of $H^0(S; \mathcal{O}(x))$, F_1, can be chosen so that its zero locus becomes an effective divisor $l_0 + (C_1 + \cdots + C_4)$. Effective divisors C_A's do not intersect with a generic element $E \in |x|$, and hence the vector bundles on E are trivial for the roots generated by C_A's. Thus, the $\mathfrak{su}(5)_{\mathrm{GUT}}$ algebra generated by C_A's ($A = 1, 2, 3, 4$) is the commutant of the $\mathrm{SU}(5)_{\mathrm{bdl}}$ structure group of vector bundles on E.

The Lie algebra of $\mathfrak{e}_8$ has 240 roots, and those of $\mathfrak{su}(5)_{\mathrm{GUT}}$ account for only 20 in the first summand of the irreducible decomposition

$$\mathbf{248} \to (\mathbf{24}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{24}) \oplus [(\mathbf{10}, \mathbf{5}) \oplus (\bar{\mathbf{5}}, \mathbf{10})] + \text{h.c.}; \tag{221}$$

[23] Curves of the form $mL_0 - \sum_I d_I L_I$ are interpreted as zero locus of a homogeneous function of degree m on $\mathbb{P}^2$ that has p_I as a zero of order d_I [27, 29].



(i) (ii)

Figure 7: Extended Dynkin diagrams of E_8. The two diagrams correspond to two different choices of Weyl chamber. By removing the node C_5 from the diagram (ii), one finds $\mathfrak{su}(5)_{\rm GUT} + \mathfrak{su}(5)_{\rm bdl}$ subalgebra generated by C_A ($A = 1, 2, 3, 4$) and $C_{\tilde{8}}$, $C_{6,7}$ and $C_{-\theta}$.

here, (R, R') denotes an irreducible component that is in R representation of $\mathfrak{su}(5)_{\rm GUT}$, and in R' of $\mathfrak{su}(5)_{\rm bdl}$. A group of roots in the $(\mathbf{10}, \mathbf{5})$ representation of $\mathfrak{su}(5)_{\rm GUT} + \mathfrak{su}(5)_{\rm bdl}$ is given by

$$C^{ab;p} = \left(\begin{array}{c} C^{ab;6^\flat} \\ C^{ab;p=6,7,8} \\ C^{ab;8^\sharp} \end{array} \right) := L_a + L_b + \left(\begin{array}{l} L_0 - (L_1 + \cdots + L_5) \\ -L_0 + L_p \\ 2L_0 - (L_1 + \cdots + L_8) \end{array} \right), \qquad (a \neq b); \qquad (222)$$

indices $1 \leq a, b \leq 5$ label five weights of the $\mathrm{SU}(5)_{\rm GUT}$ fundamental representation, and $p = \{6^\flat, 6, 7, 8, 8^\sharp\}$ (on the left hand side of (222)) five weights of the $\mathrm{SU}(5)_{\rm bdl}$ fundamental representation. It is clear that $C^{ab;p}$ form a $\mathbf{10}$ representation of $\mathfrak{su}(5)_{\rm GUT}$. One can also see that they are in the fundamental representation of $\mathfrak{su}(5)_{\rm bdl}$; the structure group is generated by four simple roots, $C_{\tilde{8}}$, C_6, C_7 and $C_{-\theta}$, where

$$\begin{aligned} C_{\tilde{8}} &= C_1 + 2C_2 + 3C_3 + 2C_4 + C_5 + 2C_8, \\ &= 2L_0 - (L_1 + L_2 + L_3 + L_4 + L_5 + L_6), \qquad (223) \\ C_{-\theta} &= -(2C_1 + 4C_2 + 6C_3 + 5C_4 + 4C_5 + 3C_6 + 2C_7 + 3C_8), \\ &= -3L_0 + (L_1 + \cdots + L_8) + L_8. \qquad (224) \end{aligned}$$

$C_{-\theta}$ corresponds to the minimal root of E_8 when C_I ($I = 1, \cdots 8$) are chosen as a set of positive simple roots, and $C_{\tilde{8}}$ is determined so that $C_{1,2,\cdots,7}$, $C_{-\theta}$ and $C_{\tilde{8}}$ form an extended Dynkin diagram of E_8 in Figure 7 (ii). Through a straightforward calculation, one can see that the five weights in (222) for a given (a, b) are obtained from $C^{ab;6^\flat}$ by applying $-C_{\tilde{8}}$, $-C_{6,7}$ and $-C_{-\theta}$ successively:

$$C^{ab;6} = C^{ab;6^\flat} - C_{\tilde{8}}, \quad C^{ab;p+1} = C^{ab;p} - C_{I=p} \quad (p = 6, 7), \quad C^{ab;8^\sharp} = C^{ab;8} - C_{-\theta}. \qquad (225)$$

Lines in I_8 that corresponds to those 10×5 roots are given by $l^{ab;p} = C^{ab;p} + x$. Those in the $(\overline{\mathbf{10}}, \bar{\mathbf{5}})$ representation are given by just multiplying (-1) to $C^{ab;p}$.

Roots in the $(\bar{\mathbf{5}}, \mathbf{10})$ representation are given by

$$C_{a;}{}^{pq} = -L_a + \begin{pmatrix} 0 & L_q & 3L_0 - (L_1 + \cdots + L_8) \\ & -2L_0 + (L_1 + \cdots + L_5) + (L_p + L_q) & L_0 - (L_6 + L_7 + L_8) + L_p \\ & & 0 \end{pmatrix}, \tag{226}$$

where $a = 1, \cdots, 5$ is the $\mathrm{SU}(5)_{\mathrm{GUT}}$ index, and $p, q \in \{6^\flat, 6, 7, 8, 8^\sharp\}$ ($p, q \in \{6, 7, 8\}$ on the right hand side). The 5×5 matrix on the right-hand side forms a $\mathbf{10}$ representation of $\mathfrak{su}(5)_{\mathrm{bdl}}$. 5×10 lines are given by $l_{a;}{}^{pq} = C_{a;}{}^{pq} + x$, and those for the $(\mathbf{5}, \overline{\mathbf{10}})$ representation are by $-C_{a;}{}^{pq} + x$.

6.1.3 SU(5) Bundle in Description B

Reference [15, 10] proposed how to describe SU(5) bundles on an elliptic fibration in terms of dP_8 fibration in description B. (See also [30].) First, the authors of [15, 10] take

$$\begin{aligned} Y^2 &= X^3 + f_0 Z^4 X + g_0 Z^6 \\ &\quad + Z' \left(a_0 Z^5 + a_2 Z^3 X + a_3 Z^2 Y + a_4 Z X^2 + a_5 XY \right) \end{aligned} \tag{227}$$

as the equation determining a del Pezzo surface S in $W\mathbb{P}^3_{1,1,2,3}$. For a del Pezzo surface $S_b = \pi_U^{-1}(b)$ ($b \in B_2$), spectral data $a_{0,2,\cdots,5}(b)$ are used in the equation above. We consider that this is a very non-trivial discovery, and we will take time in the following to examine the geometry given by this equation until this idea sinks in.

The $Z' = 0$ locus is an elliptic curve given by a Weierstrass equation $y^2 = x^3 + f_0 x + g_0$, where $(x, y) = (X/Z^2, Y/Z^3)$, which is E_b. The $Z = 0$ ($z_f = 0$) locus is an elliptic curve given by

$$y^2 - x^3 - a_5 xy = \left(y - \frac{a_5}{2} x \right)^2 - x^2 \left(x + \frac{a_5^2}{4} \right) = 0, \tag{228}$$

where now $(x, y) = (X/Z'^2, Y/Z'^3)$. Thus, this curve has a double point at $(x, y) = (0, 0)$. Moreover, one can see by examining the equation (227) around $[Z' : Z : X : Y] = [1 : 0 : 0 : 0]$ that this point is an A_4 singularity, and the $Z = 0$ locus consists of five irreducible components with their intersection form that of the extended Dynkin diagram of A_4. In the description A of del Pezzo surfaces that correspond to SU(5) bundles and unbroken $\mathfrak{su}(5)_{\mathrm{GUT}}$ symmetry, a curve given by $F_1 = 0$ also has exactly the same property. The map (214) identifies the two

curves $Z = 0$ and $F_1 = 0$ that have the same property. Thus, it is likely that the way SU(5) bundles are formulated in the description B here is the same as the one in the description A.

Among the 240 lines in I_8, 10×5 of them form a group $l^{ab;p} = C^{ab;p} + x$, but for a given $p \in \{6^\flat, 6, 7, 8, 8^\sharp\}$, the ten lines are different only by $C_{A=1,2,3,4}$ that are buried in the A_4 singularity, and they do not look different anywhere else in S. Thus, those lines can be treated as five sets of lines $l^p := l^{ab;p} \bmod C_{A=1,2,3,4}$. Reference [15, 10] provided how to describe those five "lines" in the description B: they are the zero locus of

$$FL := a_0 Z^5 + a_2 Z^3 X + a_3 Z^2 Y + a_4 Z X^2 + a_5 XY. \tag{229}$$

Suppose that a flat SU(5) bundle on an elliptic curve E_b is given by a spectral surface (144) at $b \in B_2$. The spectral surface determines five points $\{p_i\}$. Let us denote the coordinates of p_i as (x_i, y_i). Then, a map

$$l^i : \mathbb{P}^1 \ni [Z' : Z] \mapsto [Z' : Z : x_i Z^2 : y_i Z^3] \in W\mathbb{P}^3_{1,1,2,3} \tag{230}$$

is defined. The image $l^i(\mathbb{P}^1)$ falls into the zero locus of FL, and also satisfies the equation (227). Thus, each one of $l^i(\mathbb{P}^1)$'s $(i = 1, \cdots, 5)$ becomes a line in S, and is an irreducible component of the zero locus of FL. Because $FL = 0$ admits only five solutions for a given $[Z' : Z]$, those five irreducible components are all in the zero locus of FL. Those lines intersect a general element $E \in |x|$ just once, since such elements are in one to one correspondence with $\mathbb{P}^1$ parametrized by $[Z' : Z]$. The five lines $l^i(\mathbb{P}^1)$ intersect E_b at the $Z' = 0$ locus at $[1 : x_i : y_i]$, which is p_i itself [15, 10].

All those five lines pass through the A_4 singularity point. Once the A_4 singularity is blown up, then one can see explicitly that the five lines remain distinct irreducible components.

When FL is pulled back to S itself by Φ in (214), $\Phi^*(FL)$ is a global section of $\mathcal{O}(5x)$. $\Phi^*(FL)$ should be factorized into five irreducible pieces, because the $FL = 0$ locus consists of five irreducible components in the description B. Because the five lines l^p in the description A satisfy

$$\sum_{p=6^\flat}^{8^\sharp} l^p \equiv \sum_{p=6^\flat}^{8^\sharp} \left(C^{ab;p} + x\right) \equiv 5x \tag{231}$$

mod $C_{A=1,2,3,4}$, l^p's can be the irreducible zero loci of $\Phi^*(FL) \in H^0(S; \mathcal{O}(5x))$, and hence $l^i(\mathbb{P}^1)$'s in the description B.

6.1.4 Enhancement of Singularity

For a structure group smaller than $\mathrm{SU}(5)_{\mathrm{bdl}}$, the description B using $W\mathbb{P}^3_{1,1,2,3}$ is more convenient. One only needs to turn off a_5 to obtain an SU(4) bundle, and a_4 to an SU(3) bundle.

Let us consider a limit $a_5 \to 0$. It can be regarded as a limit $B_2 \ni b \to \bar{c}_V$. Then, FL is factorized:

$$FL \to Z\left(a_0 Z^4 + a_2 Z^2 X + a_3 ZY + a_4 X^2\right). \tag{232}$$

Four lines among five still scan over $\mathbb{P}^1$ parametrized by $[Z' : Z]$, but one of the five is absorbed in the $Z = 0$ locus. Suppose that the absorbed line is $l^{p=6^\flat} \equiv (L_0 - (L_1 + L_2 + L_3)) + x = C_8 + x$ mod $C_{A=1,2,3,4}$. This process adds C_8 to the set of roots whose bundle in (213) is trivial. The unbroken symmetry group is enhanced from $\mathrm{SU}(5)_{\mathrm{GUT}}$ to SO(10), because the intersection form of $C_{1,2,3,4}$ and C_8 is that of SO(10).

If a_4 is further set to zero, another line is absorbed to the $Z = 0$ locus. When the absorbed line is $l^{p=6}$ then another 2-cycle (and corresponding root)

$$l^{p=6} \equiv -L_0 + (L_1 + L_2) + L_6 + x \quad \mathrm{mod}\ C_{A=1,2,3,4}, \tag{233}$$

$$\equiv -C_5 + x \quad \mathrm{mod}\ C_{1,2,3,4,8} \tag{234}$$

joins the unbroken symmetry group, which is now E_6.

Similar process of symmetry (singularity) enhancement is observed when one of lines $l^{pq} := l_{a;}{}^{pq}$ mod $C_{A=1,2,3,4}$ is absorbed in the $Z = 0$ locus. If $l^{6^\flat 6} \equiv (-L_5 + L_6) + x$ is absorbed, $-C_5$ is now buried in the $Z = 0$ locus, and the intersection form of $C_{1,2,3,4,5}$ becomes the $(-1)\times$ Cartan matrix of SU(6).

6.2 Chirality from Four-Form Fluxes

6.2.1 From dP_8 to dP_9

A dP_8 surface S containing an elliptic curve E determines a flat bundle on E, and a dP_8 fibration $\pi_U : U \to B_2$ is able to play the same role as the spectral surface. Spectral data a_r in (227) are promoted to sections of $\mathcal{O}(rK_{B_2} + \eta)$, and the homogeneous coordinates $[Z' : Z : X : Y]$ of (227) should now be regarded as sections of

$$\mathcal{O}(J_H - \eta) \otimes \mathcal{L}_H^6, \qquad \mathcal{O}(J_H), \qquad \mathcal{O}(2J_H) \otimes \mathcal{L}_H^2, \qquad \mathcal{O}(3J_H) \otimes \mathcal{L}_H^3, \tag{235}$$

respectively.

Dual F-theory geometry is given by a dP_9 fibration[24] [12] on B_2, $\pi_W : W \to B_2$, rather than this dP_8 fibration. dP_9 (fiber) is obtained by blowing up $e_0 = [0 : 0 : c^3 : c^2]$ $(c \neq 0)$.

Such correspondence between the Heterotic and F-theories is rather a well-known story. The process of blowing up dP_8 to obtain dP_9 is well understood, and no new problems should be posed. Nevertheless, we will carefully follow this process, in order to make our presentation pedagogical, and also not to make a mistake.

In order to blow up a dP_8 surface S (to obtain a strict transform of S), we begin with blowing up the ambient space $W\mathbb{P}^3_{1,1,2,3}$. Blowing-up of $W\mathbb{P}^3_{1,1,2,3}$ is a $W\mathbb{P}^2_{1,2,3}$-fibration over $\mathbb{P}^1$. Two patches cover the new ambient space; the base $\mathbb{P}^1$ is covered by $z_f \neq \infty$ patch and $z_f \neq 0$ patch, and so is the entire ambient space. $(z_f, [Z' : X : Y])$ (resp. $(z'_f, [Z : X : Y])$) is the coordinate set in the $z_f \neq \infty$ patch (resp. $z_f \neq 0$ patch). The map to $W\mathbb{P}^3_{1,1,2,3}$ is given by $Z = Z' z_f$ from the $z_f \neq \infty$ patch and by $Z' = Z z'_f$ from the $z_f \neq 0$ patch. $z'_f = 1/z_f$. The exceptional locus σ that is mapped to the center of blow-up $e_0 = [0 : 0 : c^2 : c^3] \in W\mathbb{P}^3_{1,1,2,3}$ is given by $({}^\forall z_f, [0 : c^2 : c^3])$ in the $z_f \neq \infty$ patch (resp. $({}^\forall z'_f, [0 : c^2 : c^3])$ in the $z_f \neq 0$ patch). The defining equation of the blown-up dP_9 surface is given in the new ambient space by [30]

$$y^2 = x^3 + z_f^4 f_0 x + z_f^6 g_0 + (a_0 z_f^5 + a_2 z_f^3 x + a_3 z_f^2 y + a_4 z_f x^2 + a_5 yx), \tag{236}$$

$$y^2 = x^3 + f_0 x + g_0 + z'_f (a_0 + a_2 x + a_3 y + a_4 x^2 + a_5 yx); \tag{237}$$

the first one is in the $z_f \neq \infty$ patch, and the second one in the $z_f \neq 0$ patch. Inhomogeneous coordinates (y, x) correspond to $(Y/Z'^3, X/Z'^2)$ and $(Y/Z^3, X/Z^2)$ in the two patches, and hence they are sections of $\mathcal{O}_{\mathbb{P}^1}(3)$ and $\mathcal{O}_{\mathbb{P}^1}(2)$, respectively. A del Pezzo surface dP_9 obtained this way is an elliptic fibration on $\mathbb{P}^1$. The exceptional locus of this blow up σ passes through the infinity points, $(y, x) = (\infty, \infty)$.

Geometry of dP_9 fibration $\pi_W : W \to B_2$ is now given by the same data $a_{0,2,3,4,5}$ that described the vector bundles. It is now straightforward to cast the equation into the Weierstrass form

$$y^2 = x^3 + fx + g, \tag{238}$$

where we now use the $z_f \neq \infty$ patch. After a redefinition of the coordinates (x, y), f and g in the $z_f \neq \infty$ patch are given in $z_f = Z/Z'$ expansion as

$$f := \sum_{i=0}^{4} z_f^{4-i} f_i, \qquad g := \sum_{i=0}^{6} z_f^{6-i} g_i, \tag{239}$$

[24] The stable degeneration limit of $K3$-fibration in F-theory corresponds to the situation in Heterotic theory where the volume of the fiber T^2 is sufficiently large relatively to α'.

$$
\begin{aligned}
f_0 &= f_0, && (240)\\
f_1 &= a_2, && (241)\\
f_2 &= -\frac{1}{3}a_4^2 + \frac{1}{2}a_5 a_3, && (242)\\
f_3 &= -\frac{1}{6}a_5^2 a_4, && (243)\\
f_4 &= -\frac{1}{48}a_5^4, && (244)
\end{aligned}
$$

and

$$
\begin{aligned}
g_0 &= g_0, && (245)\\
g_1 &= a_0 - \frac{1}{3}a_4 f_0, && (246)\\
g_2 &= \frac{1}{4}a_3^2 - \frac{1}{3}a_4 a_2 - \frac{1}{12}a_5^2 f_0, && (247)\\
g_3 &= \frac{2}{27}a_4^3 - \frac{1}{6}a_5 a_4 a_3 - \frac{1}{12}a_5^2 a_2, && (248)\\
g_4 &= a_5^2\left(\frac{1}{18}a_4^2 - \frac{1}{24}a_5 a_3\right), && (249)\\
g_5 &= \frac{1}{72}a_5^4 a_4, && (250)\\
g_6 &= \frac{1}{864}a_5^6. && (251)
\end{aligned}
$$

The overall rescaling redundancy of the spectral data $[a_0 : a_2 : a_3 : a_4 : a_5]$ corresponds to the rescaling redefinition of the coordinate z_f. Apart from this rescaling, all the coefficients are determined. This precise dictionary proves very powerful later in translating the Heterotic theory description of the sheaves $\mathcal{F}_{\rho(V)}$ into F-theory language.

Now, suppose that a dP_9 surface S' is a blow up of a dP_8 surface S:

$$\pi : S' \to S. \tag{252}$$

The second cohomology group of S is generated by C_I's $(I = 1, \cdots, 8)$ of (212) and the anti-canonical divisor $x_8 = x$ of S, while that of S' by $\pi^*(C_I)$, σ and the anti-canonical divisor $x_9 = x$ of S'. Note that the anti-canonical divisors of S and S' are related via $\pi^*(x_8) = \pi^{-1}(x_8) + \sigma = x_9 + \sigma$, since x passes through the base point e_0. Note also that a topological relation

$$\pi^*(l) \sim \pi^*(C) + x_9 + \sigma \tag{253}$$

holds for a pair of $l \in I_8$ and $C \in R_8$. Intersection form among the 2-cycles of S' is given by

$$(-C_{E_8}) \oplus \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \tag{254}$$

where $\pi^*(C_I)$'s are the basis of the $(-C_{E_8})$ part and (x_9, σ) of the latter 2×2 matrix. C_{E_8} means the Cartan matrix of E_8.

6.2.2 Four-Form Fluxes

In the Heterotic string theory description, matter multiplets are characterized in terms of spectral surfaces and line bundles on them. All these pieces of information are associated with the fiber elliptic curve, which is now found in the $z'_f = 0$ ($z_f = \infty$) locus of the dP_9 fibration. On the other hand, in the F-theory description, non-Abelian gauge field of the unbroken symmetry group are localized within the locus of enhanced singularity, which is found in $z_f = 0$ locus. Chiral matter multiplets are also supposed to be at the $z_f = 0$ locus. So, how are these two descriptions related?

The spectral surface C_V in the Heterotic description only determines N points for an $\mathrm{SU}(N)$ bundle in a given elliptic fiber (which is at $z_f = \infty$), but each point corresponds to a line l^p belonging to I_8. The N lines specified by (229, 230) cover all the region of the base $\mathbb{P}^1$, including $z_f = Z/Z' = \infty$ and $z_f = 0$. Thus, in a description using dP_8 fibration (and dP_9 fibration), the information of spectral surface is not particularly localized at either end of the elliptic fibration over $\mathbb{P}^1$. In fact, the data $a_{0,2,\cdots,5}$ specifying the spectral surface controls the entire geometry of the del Pezzo fibration in (227, 236, 237).

More important in generating *chiral* matter spectrum in low-energy physics is the line bundle $\mathcal{N}_V$ on C_V, or to be more precise, γ determining $c_1(\mathcal{N}_V)$ through (9). Reference [15] introduced four-form flux $G^{(4)}_H$ in a description using dP_8 fibration, so that it plays the role of γ in the Heterotic theory. C_V is regarded locally as N copies of a local patch of B_2, and each copy corresponds to a point p_p for one of $p \in \{6^\flat, 6, 7, 8, 8^\sharp\}$ (in case of an SU(5) bundle) sweeping over B_2. γ on C_V is locally described by two forms on the each one of those copies. Suppose that a four-form flux $G^{(4)}_H$ is given in a dP_8-fibration $\pi_U : U \to B_2$. Then, γ on the copy of p_p, γ_p, is given by [15]

$$\gamma_p = \int_{l^p} i^*_{l^p} G^{(4)}_H. \tag{255}$$

Because of this correspondence between γ and $G^{(4)}_H$, only topological aspects of $G^{(4)}_H$ in dP_8 matter.

When considering SU(N) vector bundle, γ on C_V has a constraint. The vanishing of the first Chern class $c_1(V) = 0$ means that

$$\pi_{C*}\gamma = 0. \tag{256}$$

This implies that the integration of $G_H^{(4)}$ over the five lines specified by (229) should vanish. Because of the topological relation (231) satisfied by the five lines, the condition above is equivalent to

$$\int_{5x} G_H^{(4)} = 0, \qquad \text{and hence} \int_{x_8} G_H^{(4)} = 0. \tag{257}$$

Here, we assume that only SU$(5)_{\rm GUT}$ preserving fluxes are introduced in the dP_8 fibration. Because of these constraints, $G_H^{(4)}$ can be expressed as[25]

$$G_H^{(4)} \equiv \sum_{P=\tilde{8},6,7,-\theta} C_P \otimes \pi_Z^* \omega^P, \tag{258}$$

where ω^P's are 2-forms on B_2, and C_P's are 2-cycles—Poincaré dual of 2-forms—in $S = dP_8$. Fluxes proportional to x_8 should not be introduced.

We should be clear what we mean by (258). Four-form $G_H^{(4)}$ is classified by $H^4(U;\mathbb{Z})$, where $\pi_U : U \to B_2$ is a dP_8-fibration. Using Leray spectral sequence, one finds that $H^4(U;\mathbb{Z})$ has a filtration structure:

$$H^4(U;\mathbb{Z}) = F_0 \supset F_2 \supset F_4 \supset \{0\}, \tag{259}$$

with

$$F_4 \cong H^4(B_2; R^0\pi_{U*}\mathbb{Z}), \quad F_2/F_4 \cong H^2(B_2; R^2\pi_{U*}\mathbb{Z}), \quad F_0/F_2 \cong H^0(B_2; R^4\pi_{U*}\mathbb{Z}). \tag{260}$$

$G_H^{(4)}$ in (258) is understood as an element of F_2/F_4 modulo $F_4 = H^4(B_2;\mathbb{Z})$, and C_P as local generators of $R^2\pi_{U*}\mathbb{Z}$. Although the Poincare dual 2-forms of C_P's are well-defined in $H^2(U;\mathbb{Z})$ only modulo $H^2(B_2;\mathbb{Z})$, this ambiguity does not appear in (258) because $G_H^{(4)}$ is given in (258) only modulo $H^4(B_2;\mathbb{Z})$. Since x_8 is a cycle in the fiber direction, differential forms on B_2 is trivial when pulled back to x_8, and hence (257) cannot determine the F_4 part. Because of the same reason, however, (255) does not depend on the F_4 part either. Therefore, in describing vector bundles in Heterotic theory, it is sufficient to have a four-form $G_H^{(4)}$ in F_2/F_4, and leave the ambiguity in F_4 unfixed.

[25] There may or may not be an issue along the ramification locus of the spectral surface C_V. We do not address this issue in this article.

By using this explicit expression of $G_H^{(4)}$ and the intersection form

$$C^{ab;p=6^\flat,6,7,8,8^\sharp} \cdot C_{P=\tilde{8},6,7,-\theta} = \begin{pmatrix} -1 & & & & \\ 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & 1 & \end{pmatrix}, \tag{261}$$

one can see explicitly that [15]

$$\pi_{C*}(\gamma \cdot \gamma) = \sum_p \gamma_p \wedge \gamma_p = \sum_{P,Q} C_{A_4\, PQ}\omega^P \wedge \omega^Q = -\pi_{U*} G_H^{(4)} \wedge G_H^{(4)}. \tag{262}$$

Although $G_H^{(4)}$ in (258) has the ambiguity $F_4 = H^4(B_2;\mathbb{Z})$, $G_H^{(4)} \wedge G_H^{(4)}$ does not depend on the ambiguity. The same is true for other SU(N) bundles with $N < 5$; $\omega^{P=\tilde{8}}$ is set to zero for SU(4) bundles, and $\omega^{P=6} = 0$ is further imposed for SU(3) bundles.

In the F-theory compactification, there is totally an independent condition for 4-form flux $G_F^{(4)}$ on a Calabi–Yau 4-fold compactification: the (2,2) part of the four-form flux has to be primitive in order to preserve $\mathcal{N} = 1$ supersymmetry. Reference [15] observed that the condition (256) and the primitiveness condition

$$J \wedge G_F^{(4)} = 0 \tag{263}$$

are quite "similar," and certainly they are. On the other hand, $H^2(dP_9;\mathbb{Z})$ is larger than $H^2(dP_8;\mathbb{Z})$ by rank one. Thus, with only one constraint on $H^2(dP_8;\mathbb{Z})$ and one for $H^2(dP_9;\mathbb{Z})$, there should be no one-to-one correspondence between Heterotic and F-theory vacua. This gap has to be filled in order to complete the dictionary of the Heterotic–F theory duality.

The primitiveness condition (263) involves a two-form J on W and a four-form $G_F^{(4)}$ on W, where $\pi_W : W \to B_2$ is a dP_9 fibration. $H^4(W;\mathbb{Q})$, in which $G_F^{(4)}$ takes its value, has a filtration structure just like in (259):

$$H^4(W;\mathbb{Q}) = F_0 \supset F_2 \supset F_4, \tag{264}$$

with

$$F_4 \cong H^4(B_2;\mathbb{Q}), \quad F_2/F_4 \cong H^2(B_2; R^2\pi_{W*}\mathbb{Q}), \quad F_0/F_2 \cong H^0(B_2; R^4\pi_{W*}\mathbb{Q}); \tag{265}$$

notations $F_{0,2,4}$ are recycled here, as we expect little confusion. $\mathbb{Z}$ in (259) is replaced by $\mathbb{Q}$ here, because the four-form flux in F-theory is not necessarily quantized as integral value

[33]. It is known that the four-form flux in F_0 has its two legs in the T^2-fiber directions of the dP_9, and results in non SO(3,1) Lorentz symmetric vacuum [51]. Thus, we only consider $G_F^{(4)}$ that belongs to F_2 in the following. $G_F^{(4)}$ being an element of F_2 is not a sufficient condition for the SO$(3,1)$ Lorentz symmetry; we will elaborate on it later.

Similarly, the Kähler form J takes its value in $H^2(W;\mathbb{R})$, and this cohomology group also has a filtration structure:

$$H^2(W;\mathbb{R}) = E_0 \supset E_2, \quad E_2 \cong H^2(B_2;\mathbb{R}), \quad E_0/E_2 \cong H^0(B_2; R^2\pi_{W*}\mathbb{R}). \tag{266}$$

Thus, the Kähler form J is written as

$$J = \pi_W^* J_{B_2} + t_2 J_0; \tag{267}$$

projection of J into E_0/E_2 specifies a 2-form on dP_9, and J_0 is a representative of the class specified by the 2-form on dP_9. $t_2 \geq 0$ is a parameter.

In the Heterotic–F theory duality, moduli space is shared by the two theories, but one of the two theories provides a better description of some part of the moduli space, and the other of some other parts. The description in the Heterotic theory (without stringy excitations taken into account in calculations) becomes unreliable either when the Heterotic theory dilaton expectation value is large, or when the volume of the T^2 fiber becomes comparable to α'. In the first case, the base $\mathbb{P}^1$ manifold of $S' = dP_9$ has a large volume. Thus, whenever F-theory provides a better description, the volume of the base $\mathbb{P}^1$ of dP_9 is larger than that of the T^2 fiber. Therefore, in the F-theory limit, we can take it that the Kähler form on dP_9 specified by J or J_0 has a dominant contribution only from the the $\mathbb{P}^1$ base of dP_9, not from the T^2 fiber. Thus, J_0 (or J) regarded as a 2-form on dP_9 is a Poincaré dual of x_9.

The filtration structure of J and $G_F^{(4)}$ makes the analysis of the primitiveness condition (263) easier. The condition (263) takes its value in $H^6(W;\mathbb{R})$, and this group also has a filtration structure

$$H^6(W;\mathbb{R}) = G_2 \supset G_4 \supset \{0\}, \tag{268}$$

with

$$G_4 \cong H^4(B_2; R^2\pi_{W*}\mathbb{R}), \qquad G_2/G_4 \cong H^2(B_2; R^4\pi_{W*}\mathbb{R}). \tag{269}$$

We begin with the primitiveness condition in G_2/G_4, and we will come back later to the condition in the G_4 part. The G_2/G_4 part of the primitiveness condition receives contributions only from the wedge product of the E_0/E_2 part and the F_2/F_4 part, and we find that

$$t_2 x_9 \cdot G_F^{(4)} \equiv 0 \tag{270}$$

mod G_4.

The primitiveness condition (270) allows two types of local expressions for the four-form flux:

$$G^{(4)}_{F;\gamma} \equiv \sum_{I=1}^{8} C_I \otimes \omega^I, \tag{271}$$

$$G^{(4)}_{F;9} \equiv x_9 \otimes \omega^{I=9}, \tag{272}$$

here, we abuse the notation, and denote $\pi^*(C_I)$ of $H^2(dP_9;\mathbb{Z})$ as C_I, because the intersection form of $\pi^*(C_I)$'s are the same as those of C_I's in $H^2(dP_8;\mathbb{Z})$. The four-form flux $G^{(4)}_H$ (258) in the Heterotic theory description can be mapped into the first type of $G^{(4)}_F$:

$$G^{(4)}_{F;\gamma} \equiv \pi^* G^{(4)}_H; \tag{273}$$

everything is in modulo $F_4 = H^4(B_2;\mathbb{Q})$ here. We understand that the four-form flux G_γ in [10] belongs to this class modulo $F_4 = H^4(B_2;\mathbb{Q})$.

A little more attention has to paid in interpreting the other contribution (272). F-theory dual of a Heterotic compactification involves a Calabi–Yau 4-fold that is a $K3$-fibration on a base 2-fold. Although the $K3$ fiber becomes two dP_9 surfaces in the stable degeneration limit, $K3$ fiber, rather than two dP_9's, is better in understanding this aspect. As explained clearly in [51], out of 22 two-cycles of a $K3$ fiber, $2 \times 8 = 16$ two-cycles correspond to the C_I's in two dP_9's. Four-form fluxes associated with these two-cycles, like (271), satisfy the primitiveness condtion in the G_2/G_4 part. Fluxes associated with the zero section of the elliptic fibered $K3$ (like σ of dP_9) and with the T^2-fiber class (like x_9 of dP_9), on the other hand, do not either satisfy the primitiveness condition or preserve the SO(3, 1) Lorentz symmetry. Thus, such fluxes should not be introduced. Two other (1,1) two-cycles remain, and the four-form fluxes associated with these two two-cycles as well as the (2,0) and (0,2) two-cycles of $K3$ fiber correspond to the three-form fluxes of the Type IIB string theory [51]. Therefore, when the three-form fluxes are set to zero,

$$\begin{aligned}
\pi_{C*}(\gamma \wedge \gamma) &= -\pi_{U*}\left(G^{(4)}_H \wedge G^{(4)}_H\right), \\
&= -\pi_{W*}\left(G^{(4)}_{F;\gamma} \wedge G^{(4)}_{F;\gamma}\right) = -\pi_{W*}\left(G^{(4)}_F \wedge G^{(4)}_F\right).
\end{aligned} \tag{274}$$

Once again, the $F_4 = H^4(B_2;\mathbb{Z})$ ambiguity in $G^{(4)}_F$ does not matter to the relation (274).

The correspondence between the number of $M5$-branes in the Heterotic theory and the number of 3-branes in F-theory is one of the most important clues of the Heterotic–F theory

duality. The number of $M5$-branes wrapped on the elliptic fiber is given by [12]

$$n_5 = \int_{B_2} (c_2(TZ) - c_2(V_1)|_{\gamma_1=0} - c_2(V_2)|_{\gamma_2=0}) + \frac{1}{2}\gamma_1^2 + \frac{1}{2}\gamma_2^2, \tag{275}$$

where V_i and γ_i $(i = 1, 2)$ are vector bundle and discrete twisting data in (9), respectively, in the visible $(i = 1)$ and hidden $(i = 2)$ sector. The number of 3-branes in F-theory is given by [31]

$$n_3 = \frac{\chi(X)}{24} - \sum_{i=1,2} \frac{1}{2} G^{(4)}_{Fi} \wedge G^{(4)}_{Fi}, \tag{276}$$

where $F^{(3)} \wedge H^{(3)}$ contribution from the three form fluxes of the Type IIB string theory are set to zero. The equality between the first terms in n_5 and n_3 was proved in [12, 25, 32]. The equality (274) was basically shown in [15]. When the $F^{(3)} \wedge H^{(3)}$ contribution is turned on, the n_3 in F-theory will be different from the original n_5 in a Heterotic compactification (that is no longer a dual). What we did so far in section 6.2.2 is basically to collect references (mainly [15, 51]) and tell a combined story.

The extra degree of freedom in the four-form flux $G^{(4)}_F$ (the 3-form flux $F^{(3)}$ and $H^{(3)}$ in the Type IIB language) brings about another issue. In the Heterotic theory description, γ on C_V has an alternative expression

$$\gamma_p = \int_{C^p} G^{(4)}_H, \tag{277}$$

where $C^p := l^p - x_8 = C^{ab;p} \bmod C_{A=1,2,3,4}$, because the difference from (255), $x_8 \cdot G^{(4)}_H$, vanishes. In F-theory, however, the two natural guesses

$$\gamma_p = \int_{\pi^*(l^p)} G^{(4)}_F, \tag{278}$$

$$\gamma_p = \int_{\pi^*(C^p)} G^{(4)}_F \tag{279}$$

are not necessarily the same. The meaning of (278) is not even well-defined, because $\pi^*(l_p)$'s are well-defined two-cycles in dP_9, but their meaning has not been specified in $K3$. Depending on how the $\pi^*(l_p)$'s are defined in $K3$, (278) may or may not depend on the three-form fluxes $F^{(3)}$ (and $H^{(3)}$). $\pi^*(C_p)$ on the other hand, are naturally identified with one of two sets of two-cycles of $K3$ whose intersection form is $(-1)\times$ Cartan matrix of E_8. Since those two-cycles have vanishing intersection numbers with the two-cycles to which the three-form fluxes are associated with (see [51]), (279) does not depend on the choice of the extra discrete degrees

of freedom in F-theory, and is the same as (255, 277). Thus, we adopt (279) in translating γ in Heterotic theory into F-theory language. Note that γ_p's defined by (279) (and those by (278)) do not depend on the F_4 part of $G_F^{(4)}$.

6.2.3 Line Bundles on Discriminant Locus of F-theory

Here is a side remark on $\mathrm{U}(N)$ bundles and line bundles. In section 5, we studied vector bundles whose structure group is $\mathrm{U}(N)$, as well as $\mathrm{SU}(N)$ bundles. $\mathrm{U}(N)$ bundles appear in phenomenological applications of Heterotic string compactification in a form $V \cong U_N \oplus U_M$, where U_N and U_M are bundles with structure group $\mathrm{U}(N)$ and $\mathrm{U}(M)$, respectively, and $\mathrm{U}(N) \times \mathrm{U}(M) \subset \mathrm{SU}(N+M) \subset E_8$. If the bundles U_N and U_M are given by spectral cover construction on an elliptic fibered Calabi–Yau 3-fold, then "the structure group of the spectral surface" is $\mathrm{SU}(N) \times \mathrm{SU}(M)$, and the geometry of dual F-theory description has a locus of enhanced singularity that corresponds to the commutant of $\mathrm{SU}(N) \times \mathrm{SU}(M)$ in E_8. The unbroken symmetry is smaller than the commutant, because the structure group of the vector bundle V is $\mathrm{U}(N) \times \mathrm{U}(M)$, not $\mathrm{SU}(N) \times \mathrm{SU}(M)$. In the dual F-theory description, this symmetry breaking is given by a line bundle on the locus of singularity. The four-form flux determines the line bundle.

Let us see this correspondence between the Heterotic and F-theory descriptions more explicitly. We use a $V \cong U_3 \oplus U_2$ bundle with the structure group $\mathrm{U}(3) \times \mathrm{U}(2) \subset \mathrm{SU}(5)_{\mathrm{bdl}}$. Much the same story follows for a bundle $V \cong U_4 \oplus U_1$ with the structure group $\mathrm{U}(4) \times \mathrm{U}(1) \subset \mathrm{SU}(5)_{\mathrm{bdl}}$. In the spectral cover construction of the bundles U_3 and U_2, two two-forms γ_3 and γ_2 are used. Since the structure group is $\mathrm{U}(3) \times \mathrm{U}(2)$, the condition (256) does not have to be imposed separately for γ_3 and γ_2; only $\pi_{C*}(\gamma_3 + \gamma_2) = 0$ is required. Because of the overall traceless condition, $\gamma_3 + \gamma_2$ corresponds to a four-form flux of the form

$$G_F^{(4)} \equiv \sum_{P=\tilde{8},6,7,-\theta} C_P \otimes \omega^P \tag{280}$$

mod $F_4 = H^4(B_2; \mathbb{Z})$ in F-theory description. Four-form fluxes that correspond to the traceless parts of γ_3 and γ_2 are proportional to $C_{P=7,-\theta}$ and $C_{P=\tilde{8}}$, respectively, and preserve the $\mathrm{SU}(6)$ symmetry generated by $C_{A=1,2,3,4,5}$. However, a four-form flux of the form[26]

$$G_F^{(4)} \equiv \frac{1}{6}(3C_{\tilde{8}} + 6C_6 + 4C_7 + 2C_{-\theta}) \otimes \pi_{C*}\gamma_3 \equiv \frac{1}{6}(-(L_1 + \cdots + L_5) + 5L_6) \otimes \pi_{C*}\gamma_3 \tag{281}$$

[26] Coefficients of $C_{P=\tilde{8},6,7,-\theta}$ are determined so that the linear combination has vanishing intersection number with the $\mathrm{SU}(3)$ generators $C_{P=7,-\theta}$ and with the $\mathrm{SU}(2)$ generator $C_{\tilde{8}}$. The same logic is behind the choice of the linear combination coefficients in (283).

breaks the SU(6) symmetry.

In section 5.1, we studied the direct images of $\wedge^2 U_2$ of a rank-2 bundle U_2. Matter multiplets from $(\wedge^2 U_2)^{\pm 1}$ are described in terms of a line bundle $E^{\pm 1} = \mathcal{O}_{B_2}(\pm \pi_{C*}\gamma_2)$ on B_2. Remembering that the matter multiplets from $(\wedge^2 U_2)^{\pm 1}$ correspond to two-cycles $\pm C_{a;}{}^{pq=6^\flat 6} = \pm(-L_a + L_6)$, one can see that the four-form flux $G_F^{(4)}$ in (281) solely reproduces the divisor of the line bundle $E^{\pm 1}$:

$$\pm C_{a;}{}^{6^\flat 6} \cdot G_F^{(4)} = \mp \pi_{C*}\gamma_3 = \pm \pi_{C*}\gamma_2. \tag{282}$$

Note that we use the dictionary (279) here.

The dictionary (279) determines the gauge field that matter field feels in a very natural way. In F/M-theory, matter multiplets that correspond to the roots $\pm C_{a;}{}^{pq}$ (or any other roots) are fluctuations of an $M2$-brane wrapped on these two-cycles. The three-form field $C^{(3)}$ of M-theory is integrated over a two-cycle to become a gauge field for the matter field associated with the two-cycle. Degrees of freedom that appear in low-energy physics come from $M2$-branes wrapped on collapsed two-cycles. In the case of $\mathrm{U}(3) \times \mathrm{U}(2)$ bundle, the two-cycles $\pm C_{a;}{}^{6^\flat 6}$ $(a = 1, \cdots, 5)$ are also collapsed everywhere along the $z_f = 0$ locus isomorphic to B_2, a locus of A_5 singularity. Thus, the fields from those two-cycles propagate over the entire A_5 singularity locus, and are under the influence of the gauge field $\pm \int_{C_{a;}{}^{6^\flat 6}} C^{(3)}$ everywhere.

Here, bundles for roots in adjoint representation of SU(6) were obtained directly, without considering a bundle in the fundamental representation on the locus of A_5 singularity. Since we used (282), the bundles are well-defined as long as the relevant part of the four-form flux $G_F^{(4)}$ is integral. It is true that the four-form flux of F-theory has to be shifted from integral-valued quantization by $c_2(TX)/2$ [33], where X is an elliptic fibered Calabi–Yau 4-fold $\pi_X : X \to B_3$ for F-theory compactification. But at least for cases with a Heterotic dual, i.e., B_3 is a $\mathbb{P}^1$-fibration over B_2, $c_2(TX)$ can be calculated. Relevant to the issue above is a component of $c_2(TX)$ that is 2-form on B_2 and 2-form on the K3-fiber. Explicit calculation of $c_2(TX)$ reveals[27] that the terms proportional to σ and J_0 are even. Therefore, $G_F^{(4)}$ is quantized integrally for these components, and this is sufficient in guaranteeing that the bundles for fields in the roots of E_8 are well-defined.

It is also possible to turn on a line bundle in the $\mathrm{U}(1)_Y$ direction within $\mathrm{SU}(5)_{\mathrm{GUT}}$, so that the $\mathrm{SU}(5)_{\mathrm{GUT}}$ symmetry is broken down to local $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L$ and possibly global $\mathrm{U}(1)_Y$

[27] $c_2(TX) = 24\sigma \cdot J_0 - (12(K_{B_2} + t) \cdot \sigma + 46 K_{B_2} \cdot J_0) + \cdots$. Here, $t := 6K_{B_2} + \eta$, and J_0 is a divisor that corresponds to $z_f = 0$. Ellipses stand for a four-form that comes from B_2. J_0 here is the same as J_0 in [10] modulo $H^2(B_2)$. Since the coefficient of $\sigma \cdot J_0$ is even, the four-form flux $G_F^{(4)}$ can be chosen within F_2.

symmetry, as in [34]. If the line bundle in the $\mathrm{U}(1)_Y$ direction is trivial in the fiber direction in the Heterotic string compactification, then its F-theory dual exists, and a four-form flux

$$\Delta G_F^{(4)} \propto (2C_1 + 4C_2 + 6C_3 + 3C_4) \otimes \omega_Y \tag{283}$$

(modulo $H^4(B_2)$) turns on a $\mathrm{U}(1)_Y$ bundle on a locus of A_4 singularity in the F-theory dual description. Because the $\mathrm{SU}(3)_C$ and $\mathrm{SU}(2)_L$ gauge interactions come from the same worldvolume (A_4 singularity locus) in F-theory, unification of the gauge coupling constants at the Kaluza–Klein scale is maintained. The doublet–triplet splitting problem in the Higgs sector can also be solved with a $\mathrm{U}(1)_Y$ line bundles, because the spectrum in the doublet part and triplet part of $\mathrm{SU}(5)_{\mathrm{GUT}}$-$\mathbf{5}+\bar{\mathbf{5}}$ representations are different in the presence of a line bundle. Of course, a flat bundle in the $\mathrm{U}(1)_Y$ direction on an A_4 singularity locus can also maintain gauge coupling unification and solve the doublet–triplet splitting problem [35], if the A_4 singularity locus has a non-trivial fundamental group.

We have yet to study the G_4 part of the primitiveness condition (270). As long as $G_F^{(4)}$ belongs to F_2 and is further expressed as (271)[28] in F_2/F_4, then $J \wedge G_F^{(4)}$ vanish in G_2/G_4. Thus, $J \wedge G_F^{(4)}$ is contained in G_4. The four-form flux is primitive if and only if the condition in G_4 is satisfied. Reference [10] chose a representative $G_\gamma \in F_2$ from a class in F_2/F_4 specified by (271, 273). We understand that the calculations in [10] mean that

$$J \wedge (G_\gamma(+G^{(4)}_{F;B_2})) = t_2 x_9 \otimes (\bar{c}_V \cdot \gamma(+G^{(4)}_{F;B_2})) + C_I \otimes (\omega^I \wedge J_{B_2}). \tag{284}$$

Here, $G^{(4)}_{F;B_2}$ means an element of $F_4 = H^4(B_2;\mathbb{Q})$, and $(+G^{(4)}_{F;B_2})$ is added on the left-hand side, so that we can see how $J \wedge G_F^{(4)}$ changes when the representative is chosen differently. Here, we follow the way the authors of [10] specify in separating $t_2 J_0$ from $\pi_W^* J_{B_2}$. Because x_9 and C^I's remain mutually independent over the entire base 2-fold B_2, the first and second term should vanish separately in order for $G_F^{(4)}$ to be primitive (and the $\mathcal{N}=1$ supersymmetry is preserved).

The $\bar{c}_V \cdot \gamma$ term is the non-primitive contribution in [10]. This contribution, however, might be cancelled by choosing a representative in F_2 differently (put another way, by exploiting the ambiguity in $G^{(4)}_{F;B_2} \in F_4$). If the projective cylinder map [15] formulated in dP_9 fibration is used instead of the cylinder map in the determining a representative, then the "x_9 component" on the right-hand side of (284) vanishes, and at the same time, this new G_γ is odd under the involution flipping the elliptic fiber, implying that the $SO(3,1)$ Lorentz symmetry may

[28] We assume here that the $F^{(3)}$ and $H^{(3)}$ components of the four-form flux $G_F^{(4)}$ vanish.

be restored. Furthermore, in the entire $K3$-fibered Calabi–Yau 4-fold X (rather than in one of dP_9-fibred 4-fold W's), there is another contribution proportional to the T^2-fiber class x_9 from the hidden sector as well. It is more appropriate to study the primitiveness condition in the "x_9 component" in G_4 using the entire $K3$-fibred geometry. Contributions from $F^{(3)}$ and $H^{(3)}$ may or may not mix into the business. On the other hand, we need to make sure that we obtain a low-energy effective theory with $\mathrm{SO}(3,1)$ Lorentz symmetry, which means that a representative from a class in F_2/F_4 cannot be chosen arbitrarily. Although the specific choice mentioned above in the dP_9 fibration seems to be consistent with the $SO(3,1)$ Lorentz symmtery, things should be re-considered carefully in terms of $K3$-fibration once more. There is also a quantization condition on $G^{(4)}_{F;B_2}$ (possibly shifted by half integral value, depending on $c_2(TX)/2$). Therefore, we find that it is prematured to conclude that the four-form flux cannot (or can) be chosen primitive, although there is a non-primitive contribution pointed out by [10].

Reference [10] showed that the second term vanishes when C_V is irreducible. Although C_I's are independent generators of $R^2\pi_{W*}\mathbb{Z}$ locally in B_2, they are not globally over B_2 for the $C'_P s$ in the generator of the structure group of C_V. There is only one independent condition for them, and the tracelessness condition (256) guarantees that the condition is satisfied.

When the spectral surface C_V is reducible, for example, when the degree-5 cover C_V consists of irreducible degree-3 and degree-2 covers C_{U_3} and C_{U_2}, then the second term of the right-hand side of (284) consists of three independent components: one for the SU(3) part, $C_{P=7,-\theta}$, one for the SU(2) part, $C_{\tilde{8}}$ and one for the U(1) part $\propto (3C_{\tilde{8}} + 6C_6 + 4C_7 + 2C_{-\theta})$. The primitiveness condition is satisfied in the first two components, which is not more than a special case of [10]. If a $U_3 \oplus U_2$ bundle is chosen semi-stable in the Heterotic compactification, then the primitiveness condition is satisfied for the last component as well in F-theory. If the U(1) part has non-vanishing Fayet–Iliopoulos parameter, $\xi \propto \int_{B_2} \pi_{C*}\gamma_3 \wedge J_{B_2}$, then the U(1) symmetry in F-theory description also has a non-vanishing Fayet–Iliopoulos parameter $\xi_C \propto (C \cdot C_J) \int_{B_2} \omega^J \wedge J_{B_2}$, with C the two-cycle that appear in (281). This Fayet–Iliopoulos parameter is an F-theory generalization of an expression in [36] in Type IIB orientifold compactification. A related subject is discussed in [51, 10, 11].

If a line bundle in the $\mathrm{U}(1)_Y$ direction is introduced by the four-form flux in (283), the Fayet–Iliopoulos parameter may not vanish in general. In order to find a model of the real world, the four-form flux $\Delta G^{(4)}_F$ and J_{B_2} should be chosen so that the Fayet–Iliopoulos parameter vanishes. This is an F-theory translation of a condition in [34].

6.2.4 Chirality in $\rho(V) = V$

The matter curve $\bar{c}_V$ in B_2 in the Heterotic string description is where a line $l = C + x_8$ in (222) is absorbed in the $Z = 0$ locus of dP_8 fibration. In the dP_9 fibration for the F-theory description, $\pi^*(C)$ (often simply denoted in this article as C) is in the $z_f = 0$ locus, and moreover, shrinks to zero size. C is a two-cycle isomorphic to $\mathbb{P}^1$, and have self intersection number -2. Singularity of dP_9 fibration is enhanced along the matter curve because of the extra collapsed two-cycle.

Chiral matter multiplets are localized along the matter curve, and they are identified with the global holomorphic sections of sheaves $\mathcal{F}_V$ on $\bar{c}_V$. The sheaves (28) are calculated in the Heterotic string compactification. All the topological quantities associated with these sheaves should be the same in F-theory dual description, or otherwise, that is not dual. Non-topological aspects of the sheaves may be subject to corrections; we will discuss this issue in section 6.3 later.

One of the components of the divisors determining the sheaves (28) is $j^*\gamma$, γ pulled back onto the matter curve $\bar{c}_V$ from C_V. In order to have a description of matter multiplets entirely in terms of F-theory, we need a translation. It is well-known that the matter curves $\bar{c}_V$ are identified with loci of codimension-2 singularity (loci of enhanced singularity) in F-theory geometry, and the moduli space controlling the locus of the codimension-2 singularity in F-theory is the same as that of the spectral surface in Heterotic theory description. $j^*\gamma$ corresponds to (279) in F-theory. Now $j^*\gamma = \int_C i^* G_F^{(4)}$ is the field strength tensor of a gauge field obtained by integrating the 3-form field $C^{(3)}$ over the collapsed two-cycle C an $M2$-brane is wrapped on.

Now the chirality formula (36) can be rewritten in a more F-theory fashion:

$$\chi(V) = \int_{\bar{c}_V} j^*\gamma = \int_{C \times \bar{c}_V} G_F^{(4)}. \tag{285}$$

This expression would be the most natural expectation for the chirality formula, even before passing through all the calculation of direct images and translation between the Heterotic string–F-theory duality; $\chi(V)$ is obtained by counting the number of vortices that the gauge field $\int_C C^{(3)}$ creates. In this sense, this is a beautiful result, but not surprising. The true benefit of all these processes starting from the Heterotic theory is to know that codimension-3 singularities of F-theory do not give rise to extra contributions to the chirality formula. It would be difficult to say something about the codimension-3 singularities of F-theory without a better (and fundamental) formulation of F-theory itself or using duality with the Heterotic

string theory.

The chirality formula above generalizes a corresponding formula in the Type IIB string theory. In the Type IIB string theory, $SU(5)_{GUT}$ vector multiplets can be realized by wrapping five D7-branes on a holomorphic four-cycle Σ, and chiral multiplets in the **10** representation of $SU(5)_{GUT}$ are localized on a curve $\bar{c}_{\mathbf{10}}$ that is the intersection of Σ and an O7-plane. The chirality formula in the Type IIB string theory is given[29] by [38, 39, 1]

$$\chi(\mathbf{10}) = 2\int_{\bar{c}_{\mathbf{10}}} \left(\frac{F}{2\pi} - \frac{B}{(2\pi)^2\alpha'}\right). \tag{286}$$

The net chirality can also be expressed in terms of K-theory pairings [40] of D7-brane charge and O7-brane charge in a Calabi–Yau 3-fold for the Type IIB orientifold compactification [41]. But the expression in terms only of local geometry along the intersection curve allows straightforward generalization in F-theory.

Among the five two-cycles $C^{ab;p}$ for $p \in \{6^\flat, 6, 7, 8, 8^\sharp\}$ in (222) in dP_8 fibration, four are linearly independent. (Here, we ignore the difference in the choice of ab.) In the language of spectral surface, only one point in the spectra surface intersects the zero section along the matter curve generically.[30] Thus, only one out of five is absorbed in the $Z = 0$ locus at a generic point on $\bar{c}_V$. In F-theory language, this means that only on two-cycle C that corresponds to (222) collapses along the matter curve $\bar{c}_V$.

It is possible only in a local patch of B_2 to individually trace the five points $\{p_i\}_{i=1,\cdots,5}$ of the spectral surface, or five two-cycles C^p ($p = \{6^\flat, 6, 7, 8, 8^\sharp\}$). Globally on B_2, those five objects have to be glued together between two adjacent patches by the Weyl group $\mathfrak{S}_5$ of A_4. A system of five two-cycles glued by $\mathfrak{S}_5$ along B_2 is a part of $R^2\pi_{W*}\mathbb{Z}$ introduced in [15]. It will often be the case (though we do not have a proof) that there is only one topological four-cycle coming out of $H^2(B_2; R^2\pi_{W*}\mathbb{Z})$, one given by $C \times \bar{c}_V$, where C is now the collapsed two-cycle along the curve $\bar{c}_V$. The net chirality is given by the topological number of the four-form flux $G_F^{(4)}$ on this four-cycle. Only one topological number matters. This will be in one to one correspondence with the parameter λ in (178).

[29]The B-field has to be chosen half integral, if $c_1(T\Sigma)$ is not even (Freed–Witten anomaly) [37]. But, the field strength for the Type IIB open strings in the rank-2 anti-symmetric representation receives a contribution $2B$, and the vector bundle for these fields are well-defined.

[30]Two points among $\{p_i\}_{i=1,\cdots,N}$ in an elliptic fiber E_b are on the zero section only for special isolated points on the matter curves. Such exceptional points are the subject of section 6.3, and we will ignore this issue here.

6.2.5 Chirality in $\rho(V) = \wedge^2 V$

We are now ready to study the sheaf $\mathcal{F}_{\wedge^2 V}$ on the matter curve $\bar{c}_{\wedge^2 V}$, or $\widetilde{\mathcal{F}}_{\wedge^2 V}$ on the covering curves $\tilde{\bar{c}}_{\wedge^2 V}$. As we have learnt in section 5, divisors of the line bundles $\widetilde{\mathcal{F}}_{\wedge^2 V}$ always contain $\tilde{\pi}_{D*}\gamma$. Let us study what this contribution means in the F-theory language.

$\tilde{\pi}_D : D \to \tilde{\bar{c}}_{\wedge^2 V}$ is a degree-2 cover, allocating two points $\{p_i, p_j\}$ to a point in $\tilde{\bar{c}}_{\wedge^2 V}$ so that $p_i \boxplus p_j = e_0$. Let us denote the lines in I_8 for those two points (in the fundamental representation of $\mathrm{SU}(5)_{\mathrm{bdl}}$) as l^p and l^q ($p, q \in \{6^\flat, 6, 7, 8, 8^\sharp\}$); here, we consider those lines modulo C_A ($A = 1, 2, 3, 4$) for the unbroken $\mathrm{SU}(5)_{\mathrm{GUT}}$ symmetry. Now

$$\tilde{\pi}_{D*}\gamma = \int_{l^p} G^{(4)}_H + \int_{l^q} G^{(4)}_H = \int_{C^p + C^q} G^{(4)}_H = \int_{C^{pq}} G^{(4)}_H. \tag{287}$$

Here, a topological relation $C^p + C^q \equiv C^{pq}$ (mod C_A ($A = 1, 2, 3, 4$)) between the 2-cycles in (222) and (226) was used in the last equality.

There is a uniqueness problem in translating the Heterotic theory result of $\tilde{\pi}_{D*}\gamma$ into F-theory language, as we encountered in translating $j^*\gamma$ into (278) or (279). We adopt

$$\tilde{\pi}_{D*}\gamma = \int_{\pi^*(C^{pq})} G^{(4)}_F, \tag{288}$$

in the same spirit as we chose (279) for $j^*\gamma$. This is the field strength of a gauge field obtained by integrating the 3-form field $C^{(3)}$ over the two-cycle $C_{a;}{}^{pq}$—a gauge field an $M2$-brane wrapped on the collapsed two-cycle $C_{a;}{}^{pq}$ is coupled to. As long as we adopt this rule of translation, the flux quanta associated with the non-E_8 part of the two-cycles in $K3$-fiber do not have an influence on the net chirality, or even on γ that describes a line bundle in F-theory.

It is interesting to note that the notion of the covering curve $\tilde{\bar{c}}_{\wedge^2 V}$ we introduced in sections 4 and 5 is not only for mathematical convenience. An $M2$-brane wrapped on a cycle $\pi^*(C^{pq})$ propagates on the covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$, not on the matter curve $\bar{c}_{\wedge^2 V}$, because the each point of the covering mater curve is in one to one correspondence with the collapsed two-cycle.

The chirality formula in this (pair of) irreducible representation(s) follows immediately:

$$\chi(\wedge^2 V) = \int_{\tilde{\bar{c}}_{\wedge^2 V}} \tilde{\pi}_{D*}\gamma = \int_{C^{pq} \times \tilde{\bar{c}}_{\wedge^2 V}} G^{(4)}_F. \tag{289}$$

This is quite a natural result, once again. But all the hard work in section 5 that has led to this conclusion tells us that we do not need to add an extra contributions associated

with codimension-3 singularities of F-theory; it was the part hardly accessible with limited intuition in F-theory, yet our study using the Heterotic–F theory duality shows that (289) is indeed fine.

This expression is an F-theory generalization of the Type IIB chirality formula in a corresponding system. Here, we imagine a Type IIB set up where five D7-branes are wrapped on a holomorphic four-cycle $\Sigma_{\mathbf{5}}$ of a Calabi–Yau 3-fold, and another D7-brane on another four-cycle $\Sigma_{\mathbf{1}}$. Topological U(1) gauge field configuration F_5 and F_1 is assumed on the both four-cycles, $\Sigma_{\mathbf{5}}$ and $\Sigma_{\mathbf{1}}$, respectively. Then, the net chirality in the $\mathrm{SU}(5)_{\mathrm{GUT}}$-$\bar{\mathbf{5}}$ representation is given by [39]:

$$\#(\bar{\mathbf{5}}, \mathbf{1}^+) - \#(\mathbf{5}, \mathbf{1}^-) = \int_{\Sigma_{\mathbf{5}} \cdot \Sigma_{\mathbf{1}}} i^* \left(\frac{F_1}{2\pi} \right) - i^* \left(\frac{F_5}{2\pi} \right). \tag{290}$$

This expression, written only in terms of local geometry around the D7–D7 intersection curve, is equivalent to the one in [42] given by pairing of D-brane charge vectors in K-theory [40, 43, 44]. The F-theory formula (289) is the most natural generalization of the local formula of the Type IIB string theory (290).

6.2.6 Chirality in $\rho(V) = \wedge^3 V$

It is now straightforward to provide an F-theory interpretation for the $\tilde{\pi}_{T\pm *}\gamma|_{T_\pm}$ contribution to the sheaves $\widetilde{\mathcal{F}}_{\wedge^3 V \pm}$ in (204). In the Heterotic theory description,

$$\tilde{\pi}_{T_\pm *}\gamma = \int_{l^p + l^q + l^r} G_H^{(4)} = \int_{C^p + C^q + C^r} G_H^{(4)} = \int_{C^{pqr}} G_H^{(4)}, \tag{291}$$

where C^{pqr}'s are now two-cycles that correspond to the roots in the $(\wedge^3 V, \mathbf{1}, \mathbf{2})$ of the group $\mathrm{SU}(6) \times \mathrm{SU}(3) \times \mathrm{SU}(2) \subset E_8$. In F-theory, this is replaced by $\int_{\pi^*(C^{pqr})} G_F^{(4)}$.

In the SU(6)-bundle compactification of the Heterotic string theory, there are two types of massless chiral multiplets in the $(\mathbf{1}, \mathbf{2})$ representation of the unbroken symmetry group SU(3)×SU(2). One group of multiplets is $H^0(\bar{c}_{\wedge^3 V}; \nu_* \widetilde{\mathcal{F}}_{\wedge^3 V +})$, and the other $H^0(\bar{c}_{\wedge^3 V}; \nu_* \widetilde{\mathcal{F}}_{\wedge^3 V -}) \simeq [H^1(\bar{c}_{\wedge^3 V}; \widetilde{\mathcal{F}}_{\wedge^3 V +})]^\times$. Thus, a net chirality can be defined in the SU(2)-doublet sector as the difference between the degrees of freedom of the two groups. It is

$$\chi(\wedge^3 V)_+ := \chi(\bar{c}_{\wedge^3 V}; \nu_* \widetilde{\mathcal{F}}_{\wedge^3 V +}) = T_+ \cdot \gamma = -\chi(\bar{c}_{\wedge^3 V}; \nu_* \widetilde{\mathcal{F}}_{\wedge^3 V -}). \tag{292}$$

In F-theory, this chirality is given by

$$\chi(\wedge^3 V)_+ = \int_{C^{pqr} \times \bar{c}_{\wedge^3 V}} G_F^{(4)}. \tag{293}$$

6.3 Codimension-3 Singularities in F-theory Geometry

There are many aspects in low-energy physics that do not depend only on the net chirality in each representation. One will be surely interested in whether the two Higgs doublets of the Minimal Supersymmetric Standard Model can be vector-like in nature. If there are light vector-like $\mathrm{SU}(5)_{\mathrm{GUT}}$-charged multiplets, they may serve as messenger sector of gauge mediated supersymmetry breaking, for example. For these purposes, we need to know both $h^0(\bar{c}_{\wedge^2 V}; \mathcal{F}_{\wedge^2 V})$ and $h^1(\bar{c}_{\wedge^2 V}; \mathcal{F}_{\wedge^2 V})$ separately, not just the difference between these two numbers. Even if there are no vector-like pairs of multiplets in low energy, heavy vector-like states can make some qualitative differences in physics observed in low energy (e.g. [24]). However, $d = \deg c_1(\widetilde{\mathcal{F}}_{\rho(V)})$ alone cannot determine both h^0 and h^1, if $0 \leq d \leq 2g-2$, where g is the genus of $\tilde{\bar{c}}_{\rho(V)}$ [18]. Since g is generically large for the curve $\tilde{\bar{c}}_{\wedge^2 V}$ (see Tables 2, 3), and d is not different very much from $g-1$ (c.f. (177)), d is quite likely to be in the window above, indeed. More numerical information such as values of Yukawa couplings of quarks and leptons depend on more detail of the divisors specifying the sheaves on the matter curves; Yukawa couplings may depend on the values of global holomorphic sections at intersection points of matter curves [11], and just d is clearly not enough information in determining the values of the sections.

In order to obtain all this information, one needs to use all the information of the line bundles $\tilde{\mathcal{F}}_{\rho(V)}$ on the matter curves, or of the divisors that determine them. We have clarified how the divisors proportional to γ originate in F-theory geometry. It is now time to do the rest. All the divisors of $\tilde{\mathcal{F}}_{\rho(V)}$'s contain a pullback of the canonical divisor of the base manifold B_2. Since the Heterotic and F-theory share the same base 2-fold B_2, K_{B_2} is well-defined in F-theory as well. Thus, we study the rest of the components of the divisors on the (covering) matter curves in this subsection. We will see that most of the divisors that we identified in section 5 in Heterotic theory compactification correspond to codimension-3 singularities in F-theory geometry.

Geometry of dP_9 fibration $\pi_W : W \to B_2$ is specified by equations (236, 237), and (singular) geometry along $z_f = 0$ locus is described better by (236). a_r in this equation are global holomorphic sections of $\mathcal{O}(rK_{B_2} + \eta)$ on B_2, and they originally described the spectral surfaces. Parameters a_i^{Tate} that appeared in [13] are related to these a_r^{SS} (SS is a short hand notation of spectral surface) are related through

$$a_{6-i}^{\mathrm{Tate}} = (-1)^i a_i^{\mathrm{SS}} z_f^{5-i}, \tag{294}$$

and the property

$$\text{ord } a_r^{\text{Tate}} = r - 1 \tag{295}$$

required for geometry with A_4 singularity is reproduced from the correspondence above. Note in the Heterotic–F theory dictionary in (240–251), however, that all the coefficients are already fixed except the rescaling of the coordinate z_f. Codimension-3 singularities of F-theory geometry were studied in detail in [32]. (See also [45].) We will use the precisely determined dictionary (240–251) instead and do the same calculation over again in the following.

6.3.1 Geometry with a Locus of E_6 Singularity

If the sections a_4 and a_5 vanish, then the dP_9-fibered geometry develops a locus of E_6-type singularity at $z_f = 0$:

$$y^2 = x^3 + g_2 z_f^4. \tag{296}$$

The discriminant of the elliptic fibration is given by

$$\Delta = z_f^8 \left(\frac{27}{16} a_3^4 + \frac{z_f}{2} (8a_2^3 + 27a_3^2 a_0) + z_f^2 (27a_0^2 + \cdots) + \cdots \right). \tag{297}$$

The sheaves $\mathcal{F}_V$ and $\mathcal{F}_{\wedge^2 V}$ on the matter curve $\bar{c}_V$ involve a divisor $b^{(a)} = j^* r$. In order to obtain a description of these sheaves in F-theory, we would not want these divisors to be expressed in terms of the ramification divisor r, which is rather closely associated with geometry of vector bundles in Heterotic theory. In section 5.2,

$$b^{(a)} := \text{div } a_2 \tag{298}$$

was the definition of the divisor on $\bar{c}_V$. Since the defining equation of dP_9-fibered geometry of F-theory uses the same data a_0, a_2 and a_3, we know where the support of $b^{(a)}$ is in F-theory geometry as well.

The codimension-1 $z_f = 0$ locus in the base 3-fold is now a zero of the discriminant Δ of order z_f^8. The matter curve $\bar{c}_V$ is a codimension-2 locus in the 3-fold and $\Delta \sim \mathcal{O}(z_f^9)$ there. Singularity is enhanced from E_6 to E_7 there. Because the coefficient of z_f^9 term is $4a_2^3$ when it is evaluated on the matter curve $a_3 = 0$, $b^{(a)}$ is actually the codimension-3 locus in the 3-fold where $\Delta \sim \mathcal{O}(z_f^{10})$. Singularity is now enhanced to E_8. Thus, the divisor $b^{(a)}$ on the matter curve can be defined as the codimension-3 singularity of the dP_9-fibered geometry of F-theory.

We are now able to describe the sheaves (28) and (85) entirely in terms of geometric object in F-theory. We lack an explanation for why the coefficient of the divisor $b^{(a)}$ is 1/2, nothing else. Of course we know that it has to be 1/2, because otherwise,

$$\deg \ \left(K_{B_2} + c_a b^{(a)}\right) = \frac{1}{2}\deg \, K_{\bar{c}_V} + \left(c_a - \frac{1}{2}\right) \deg \, b^{(a)}, \tag{299}$$

and a consistency relation $\mathcal{F}_V \cong K_{\bar{c}_V} \otimes \mathcal{F}^{-1}_{V^\times}$ no longer holds for $c_a \neq 1/2$. We believe that there must be an explanation for 1/2 in terms of local geometry around the codimension-3 singularity within F-theory itself, not just from a global consistency above, but we do not have one; the coefficient was determined through the Heterotic–F theory duality, instead. For practical purposes such as model building, one can just use the coefficient 1/2, and there is nothing wrong. A local explanation of 1/2 in F-theory itself remains an (academic but quite interesting) open problem for the future.[31]

6.3.2 Geometry with a Locus of $E_5 = D_5$ Singularity

If only the global holomorphic section a_5 vanishes, and $a_{0,2,3,4}$ are generic, then we have a locus of D_5 singularity.

$$y^2 = x^3 + f_2 z_f^2 x + g_3 z_f^3. \tag{300}$$

The discriminant is given by

$$\Delta = z_f^7 \left(a_4^3 a_3^2 + z_f \left(\frac{27}{16} a_3^4 - \frac{9}{2} a_3^2 a_2 a_4 - a_4^2 (a_2^2 - 4 a_0 a_4) \right) + \mathcal{O}(z_f^2) \right). \tag{301}$$

Singularity is D_5 along the codimension-1 $z_f = 0$ locus, and $\Delta \sim \mathcal{O}(z_f^7)$. Along codimension-2 locus $a_4 = 0$ ($\bar{c}_V$) and $a_3 = 0$ ($\bar{c}_{\wedge^2 V}$), $\Delta \sim \mathcal{O}(z_f^8)$, and the singularity is enhanced to E_6 and D_6, respectively.

The description of $\mathcal{F}_V$ involves a divisor $j^* r$ on $\bar{c}_V$, and that of $\tilde{\mathcal{F}}_{\wedge^2 V}$ a divisor $\tilde{b}^{(c)}$ on $\tilde{\bar{c}}_{\wedge^2 V}$ [resp. $b^{(c)}$ on $\bar{c}_{\wedge^2 V}$]. Thus, let us think of characterizing those divisors in terms of F-theory geometry.

Because the argument around (97) is valid independent of rank N of the vector bundles in Heterotic theory compactification, the relation (96) holds for any N. Here, the definition of $b^{(a)}$ is now

$$b^{(a)} := \text{div} \ a_{N-1} \tag{302}$$

[31](note in v.4) A clear answer is now given to this problem; see section 5 of [52].

on the matter curve $\bar{c}_V$ ($a_N = 0$). The E_6 singularity along the matter curve $a_4 = 0$ is enhanced to E_7 at the codimension-3 singularity, $a_3 = 0$, and this is where we find the divisor $b^{(a)}$. Thus, this can be used as the F-theory characterization of the divisor $b^{(a)}$. At such codimension-3 singularities, the order of the discriminant Δ is enhanced, an extra two-cycle collapses, and sometimes, multiple two-cycles exhibit a monodromy around a codimension-3 singularity. Such nature of F-theory geometry may be able to account for the coefficient 1/2 of the divisor $b^{(a)}$, but we do not have a clear answer for this problem, apart from the global consistency condition we mentioned after (299).

The divisor $\tilde{b}^{(c)}$ is where the covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$ is ramified over the matter curve $\bar{c}_{\wedge^2 V}$, and the branched locus was characterized as the zero locus of $R^{(4)}$; see (104) and (105). Since the coefficient of the z_f^8 term of the discriminant is

$$a_4^2 R^{(4)} + \mathcal{O}(a_3^2), \tag{303}$$

it is at $b^{(c)}$ that this coefficient vanishes,[32] and $\Delta \sim \mathcal{O}(z_f^9)$.

It is interesting to note that there is a contribution $\tilde{b}^{(c)}$ to the divisor of $\tilde{\mathcal{F}}_{\wedge^2 V}$, but from the other codimension-3 singularity such as $b^{(a)}$ which also defines a divisor on $\bar{c}_{\wedge^2 V}$. On the other hand, the divisor $b^{(a)}$ contributes to $\mathcal{F}_V$ with a coefficient 1/2. We regret that we only have the results, and do not have a local explanation for these interesting phenomena, apart from the global consistency conditions such as $\tilde{\mathcal{F}}_{\wedge^2 V} \cong K_{\tilde{\bar{c}}_{\wedge^2 V}} \otimes \tilde{\mathcal{F}}^{-1}_{\wedge^2 V}$ (or equivalently (140)).

6.3.3 Geometry with a Locus of $E_4 = A_4$ Singularity

For fully generic choice of $a_{0,2,3,4,5}$, a locus of A_4 singularity exists in the $z_f = 0$ locus. The singularity is at $(x, y) = (0, 0)$ in the coordinate used in (236), and $(x, y) = (a_5^2/12, 0)$ in the coordinates for the Weierstrass-form equation in (238). The discriminant around the A_4

[32] It was reported in [32] that the coefficient of the $\mathcal{O}(z_f^8)$ term vanishes on $\bar{c}_{\wedge^2 V}$ when

$$R \propto -\frac{3}{4} f_1^2 + 2 g_1 h + 3 f_0 h^2 = 0. \tag{304}$$

There is a loose Heterotic–F-theory correspondence between sections of a common line bundle:

$$f_1, a_2 \in \Gamma(B_2; \mathcal{O}(2K_{B_2} + \eta)), \qquad g_1, a_0 \in \Gamma(B_2; \mathcal{O}(\eta)), \tag{305}$$
$$h, a_4 \in \Gamma(B_2; \mathcal{O}(4K_{B_2} + \eta)), \qquad f_0 \in \Gamma(B_2; \mathcal{O}(-4K_{B_2})). \tag{306}$$

It is only with the precise Heterotic–F dictionary (240–251), however, that one can find (or even discuss) an agreement between the divisors of $R^1 \pi_{Z*} \wedge^2 V$ in the Heterotic theory and the codimension-3 singularities in F-theory. Note that f_0 responsible for the complex structure of the elliptic fiber enters also in (246, 247).

singularity locus is given by

$$\Delta = z_f^5 \left(\frac{1}{16} a_5^4 P^{(5)} + \frac{z_f}{16} a_5^2 \left(12 a_4 P^{(5)} - a_5^2 R^{(5)}\right) \right.$$
$$\left. + z_f^2 \left(a_3^2 a_4^3 + \mathcal{O}(a_5)\right) + \mathcal{O}(z_f^3)\right). \tag{307}$$

The matter curve $\bar{c}_V$ is given by $a_5 = 0$, and $\bar{c}_{\wedge^2 V}$ by $P^{(5)} = 0$, where $P^{(5)}$ is defined in (146).

The sheaf $\mathcal{F}_V$ on the matter curve $\bar{c}_V$ involves a divisor $j^* r = b^{(a)}$, which is identified with the locus of $a_5 = a_4 = 0$, just like we argued for the case with D_5 singularity. This locus corresponds to the type (a) intersection points of $\bar{c}_V$ and $\bar{c}_{\wedge^2 V}$ (see Figure 4). Singularity is enhanced here, and $\Delta \sim \mathcal{O}(z_f^8)$.

The sheaf $\tilde{\mathcal{F}}_{\wedge^2 V}$ on the covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$ involves $b^{(c)}$. Among the type (c) ramification points, we have identified the locus of type (c1) points, and their positions on $\bar{c}_{\wedge^2 V}$ was specified by the zero locus of $R^{(5)}$ defined in (150). One can see from (307) that this is exactly the place in F-theory geometry where singularity is enhanced, and the discriminant becomes $\Delta \sim \mathcal{O}(z_f^7)$ from $\Delta \sim \mathcal{O}(z_f^6)$ on generic points on $\bar{c}_{\wedge^2 V}$.

Although the divisors of the sheaves $\mathcal{F}_V$ and $\mathcal{F}_{\wedge^2 V}$ correspond to codimension-3 singularities of F-theory geometry, however, not all those singularities contribute to the divisors, just like we have already seen in the case of D_5 singularity locus. For example, along the matter curve $\bar{c}_V$, $\Delta \sim \mathcal{O}(z_f^8)$ at type (d) intersection points of $\bar{c}_V \cdot \bar{c}_{\wedge^2 V}$ as well ($a_3 = 0$ as well as $a_5 = 0$), but there is not contribution to the divisor of $\mathcal{F}_V$ there. Similarly, there is no contribution at the type (a) intersection points to $\mathcal{F}_{\wedge^2 V}$, although $\Delta \sim \mathcal{O}(z_f^7)$ at $a_5 = 0$ along $\bar{c}_{\wedge^2 V}$. We do not have an explanation which codimension-3 singularities contribute by how much, apart from the global consistency conditions $\mathcal{F}_V \cong K_{\bar{c}_V} \otimes \mathcal{F}_{V^\times}^{-1}$ and $\tilde{\mathcal{F}}_{\wedge^2 V} \cong K_{\tilde{\bar{c}}_{\wedge^2 V}} \otimes \tilde{\mathcal{F}}_{\wedge^2 V}^{-1}$ (or equivalently (156)).

7 Describing Matter Multiplets in F-theory

F-theory is compactified on an elliptic Calabi–Yau 4-fold

$$\pi_X : X \to B_3 \tag{308}$$

in order to obtain low-energy effective theory with $\mathcal{N} = 1$ supersymmetry. Suppose that the elliptic fibration is given by

$$y^2 = x^3 + f x + g, \tag{309}$$

where f and g are global holomorphic sections of line bundles $\mathcal{L}_F^{\otimes 4}$ and $\mathcal{L}_F^{\otimes 6}$, respectively. Calabi–Yau condition of X requires that $\mathcal{L}_F \simeq \mathcal{O}(-K_{B_3})$.

The discriminant locus of the elliptic fibration is given by

$$\mathrm{div}\,\Delta = -12K_{B_3}. \tag{310}$$

Suppose that the discriminant locus has an irreducible component S with multiplicity c:

$$\mathrm{div}\,\Delta = c\Sigma + \cdots . \tag{311}$$

$c = 5$ when X has A_4 singularity along Σ, $c = 7$ for D_5 singularity and $c = 8$ for E_6 singularity. A topological class of divisor η on Σ is defined by [46]

$$N_{\Sigma|B_3} = \mathcal{O}_\Sigma(6K_\Sigma + \eta), \tag{312}$$

where K_Σ is the canonical divisor of Σ, and $N_{\Sigma|B_3}$ is the normal bundle of $\Sigma \hookrightarrow B_3$. Normal coordinate of Σ in B_3, z_f, is a section of the normal bundle.

Global holomorphic sections on B_3, f and g, can be expressed around Σ by expansion in the normal coordinate z_f.

$$f = \sum_{i=0} z_f^i f_{4-i}, \tag{313}$$

$$g = \sum_{i=0} z_f^i g_{6-i}. \tag{314}$$

Because

$$-\left.K_{B_3}\right|_\Sigma = -K_\Sigma + N_{\Sigma|B_3} = 5K_\Sigma + \eta, \tag{315}$$

f_{4-i} and g_{6-i} are holomorphic sections of the following:

$$f_{4-i} \in \Gamma(\Sigma; \mathcal{O}(4(5K_\Sigma + \eta)) \otimes N_{\Sigma|B_3}^{-i}) = \Gamma(\Sigma; \mathcal{O}(20-6i)K_\Sigma + (4-i)\eta), \tag{316}$$

$$g_{6-i} \in \Gamma(\Sigma; \mathcal{O}(6(5K_\Sigma + \eta)) \otimes N_{\Sigma|B_3}^{-i}) = \Gamma(\Sigma; \mathcal{O}(6(5-i)K_\Sigma + (6-i)\eta)). \tag{317}$$

In order to preserve A_4, D_5 or E_6 singularity along Σ, there should exist global holomorphic sections

$$a_r \in \Gamma(\Sigma; \mathcal{O}(rK_\Sigma + \eta)), \qquad r = 0, 2, 3, 4, 5, \tag{318}$$

so that f_{4-i}'s and g_{6-i}'s are given globally on Σ as in (240–251). Note that a_r's and the divisor η are characterized only in terms of geometry of X around Σ, and one does not need

to refer to a dual description in the Heterotic string theory, or even to assume that B_3 is a $\mathbb{P}^1$ fibration over Σ and a Heterotic dual exists.

Matter curves are determined by the sections $a_{0,2,\cdots,5}$ on Σ, and various divisors on the matter curves by locus of codimension-3 singularities. Three-form background configuration influences the sheaves $\mathcal{F}_{\rho(V)}$ only through its behavior on the collapsed two-cycles along Σ or matter curves on it. Thus, the sheaves $\mathcal{F}_{\rho(V)}$ are described only in terms of local geometry around Σ.

Since only local information of X around Σ is involved, descriptions of $\mathcal{F}_{\rho(V)}$ still hold true, as long as local geometry remains the same. In particular, the same description of the sheaves can be used even when the Heterotic dual does not exist. It is true that the calculation of $R^1\pi_{Z*}\rho(V)$ and hence $\tilde{\mathcal{F}}_{\rho(V)}$ that we carried out is reliable only in the region of the moduli space where the volume of T^2 is reasonably larger than α' (but the stability of the bundle given by spectral cover construction is guaranteed only when the size of T^2 fiber is smaller than the typical size of the base manifold B_2). Such large T^2 region of the moduli space corresponds to the stable degeneration limit of a K3-fiber into two dP_9-fibration, one for the visible sector E_8, and the other for the hidden sector E_8. As the volume of T^2 becomes comparable to α', spectral surface parameters $[a_0 : a_2 : a_3 : a_4 : a_5] \in \mathbb{P}^4$ describing four Wilson lines in the T^2 fiber directions become a part of the $\mathrm{SO}(18,2)/\mathrm{SO}(18)\times\mathrm{SO}(2)$ Narain moduli of the T^2 compactification of the Heterotic string theory, and should be treated in a way mixed up with the Kähler and complex moduli parameters of T^2. Although field theory calculation in the Heterotic theory is unreliable and the meaning of $a_{0,2,3,4,5}$ is not clear, we can rather take (240–251) as the definition of $a_{0,\cdots,5}$ in such region of the moduli space. An idea that the divisors specifying the sheaves $\mathcal{F}_{\rho(V)}$ are associated with codimension-3 singularities seems so natural (at least to the authors) that we speculate that the relations between the sheaves and the codimension-3 singularities persist without a correction in the entire region of the F-theory moduli space. We obtain $\mathcal{F}_{\rho(V)}$ for generic configuration of F-theory geometry X in this way. We will be more explicit for specific cases later on in this section.

Since the description of $\mathcal{F}_{\rho(V)}$ determined as above relies only on local geometry of X along Σ, such a set up may allow for local model building of particle physics; "local" in the sense that the geometry in the other parts of X does not matter (very much) to particle physics of the visible sector. That will be an F-theory version of [47] in the Type IIB string, [48] in Type IIA and [49] in G_2 holonomy compactification of eleven-dimensional supergravity.

7.1 On an E_6-Singularity Locus

The Calabi–Yau 4-fold X develops a locus of E_6 singularity along Σ, when a_4 and a_5 are set to zero. The discriminant of the elliptic fibration (308) is given by (297) around Σ. Zero locus of a_3 defines a matter curve[33] $\bar{c}_{(\mathbf{27})}$, along which the singularity in the directions transverse to Σ becomes E_7. There is an extra collapsed two-cycle along the matter curve, so that the intersection form of the $\mathbb{P}^1$'s becomes E_7. The matter curve $\bar{c}_{(\mathbf{27})}$ belongs to a topological class $|3K_\Sigma + \eta|$. The singularity is enhanced even to E_8 at some special points on the matter curve, determined by the condition $a_2 = 0$. Collection of these points define a divisor $b^{(a)}$ on $\bar{c}_{(\mathbf{27})}$ as in (302).

Suppose that a vector bundle $\mathcal{E}$ is turned on the E_6 singular locus Σ. Then, the unbroken symmetry at low energy is a subgroup H of E_6 that commutes with the structure group of $\mathcal{E}$. First group of chiral multiplets in low energy effective theory arises from the entire bulk of Σ. By generalizing B_2 in (44, 46) to a general E_6 discriminant locus Σ of F-theory, the matter multiplets are in [10, 11]

$$H^1(\Sigma; \mathrm{adj.}(\mathcal{E})) \oplus H^0(\Sigma; \mathcal{O}(K_\Sigma) \otimes \mathrm{adj.}(\mathcal{E})). \tag{319}$$

References [10, 11] build an intrinsic formulation of F-theory itself and explain why the latter cohomology group is for a bundle involving K_Σ, rather than $N_{\Sigma|B_3}$. Calculation in Heterotic dual also concludes that K_Σ should be used, rather than $N_{\Sigma|B_3}$, regardless of whether the discriminant locus Σ has codimension-2 (and -3) loci of enhanced singularity or not [10]. The Heterotic theory calculation in section 3 suggests, however, that not all the generators of the cohomology group $H^0(\Sigma; \mathcal{O}(K_\Sigma) \otimes \mathrm{adj.}(\mathcal{E}))$ are massless in fact. Only the kernel and cokernel of the map (43) remain massless there, and it may be that similar phenomenon exists in F-theory. The formula for the net chirality itself does not depend on this subtlety, however, and a generalization of (53) gives

$$\chi(R_H) = -\int_\Sigma c_1(T\Sigma) \wedge c_1(\rho(\mathcal{E})), \tag{320}$$

just like in [1, 11]; here, we assume that the structure group of $\mathcal{E}$ is a proper subgroup of E_6 and its commutant is H, and the net chirality is considered for a pair of Hermitian conjugate pair of irreducible representations $(\rho(\mathcal{E}), R_H) + (\rho(\mathcal{E})^\times, R_H^\times)$.

[33] When we use representations of unbroken symmetries (like $\mathbf{27}$ of E_6) as subscripts, instead of representations of structure groups, we will do so by using parenthesis, like $(\mathbf{27})$.

The second group of chiral multiplets are localized on the matter curve $\bar{c}_{(\mathbf{27})}$.

$$H^0\left(\bar{c}_{(\mathbf{27})};\mathcal{O}\left(i^*K_\Sigma+\frac{1}{2}b^{(a)}\right)\otimes\mathcal{L}_G\otimes\rho_{(\mathbf{27})}(\mathcal{E})\right), \tag{321}$$

$$H^0\left(\bar{c}_{(\mathbf{27})};\mathcal{O}\left(i^*K_\Sigma+\frac{1}{2}b^{(a)}\right)\otimes\mathcal{L}_G^{-1}\otimes\rho_{(\overline{\mathbf{27}})}(\mathcal{E})\right). \tag{322}$$

where $\mathcal{L}_G$ is a line bundle on $\bar{c}_{(\mathbf{27})}$ determined by a gauge field obtained by integrating the 3-form field on the vanishing 2-cycle along $\bar{c}_{(\mathbf{27})}$. See (279). The formula for the net chirality is given by $\chi=\int_{\bar{c}_{(\mathbf{27})}}c_1(\mathcal{L}_G\otimes\rho_{(\mathbf{27})}(\mathcal{E}))$, or simply by (285) in the absence of the bundle $\mathcal{E}$ on the locus of E_6 singularity.

7.2 On a $D_5=E_5$-Singularity Locus

If a_5 is set to zero and $a_{0,2,3,4}$ do not vanish, then a locus of $E_5=D_5$ singularity develops along Σ. The discriminant Δ is given by (301) around Σ.

As for the chiral matter multiplets arising from the entire worldvolume of the D_5 singularity locus Σ, everything stated in the second paragraph of section (7.1) holds true, after replacing E_6 by SO(10), and interpret $\mathcal{E}$ as a bundle in SO(10).

The zero locus of a_4 defines a matter curve $\bar{c}_{(\mathbf{16})}$, and singularity of X in the direction transverse to Σ becomes E_6. An extra two-cycle is collapsed along this matter curve, so that the intersection form becomes E_6. The singularity becomes E_7 on special points on $\bar{c}_{(\mathbf{16})}$, specified by $a_3=0$. These points define a divisor $b^{(a)}$ as in (302). Chiral multiplets localized on the matter curve are

$$H^0\left(\bar{c}_{(\mathbf{16})};\mathcal{O}\left(i^*K_\Sigma+\frac{1}{2}b^{(a)}\right)\otimes\mathcal{L}_G\otimes\rho_{(\mathbf{16})}(\mathcal{E})\right), \tag{323}$$

$$H^0\left(\bar{c}_{(\mathbf{16})};\mathcal{O}\left(i^*K_\Sigma+\frac{1}{2}b^{(a)}\right)\otimes\mathcal{L}_G^{-1}\otimes\rho_{(\overline{\mathbf{16}})}(\mathcal{E})\right), \tag{324}$$

where $\mathcal{L}_G$ is a line bundle determined by a 2-form on $\bar{c}_{(\mathbf{16})}$ which is obtained by integrating the four-form field strength $G_F^{(4)}$ over the two-cycle collapsed along $\bar{c}_{(\mathbf{16})}$. The net chirality is given by (285), if $\mathcal{E}$ is trivial.

Another group of chiral multiplets arises from another matter curve $\bar{c}_{(\mathbf{vec})}$, which is given by zero locus of a_3. Singularity of X becomes D_6 along this curve. There are two two-cycles collapsing along this curve. We are already familiar with this phenomenon in the Type IIB string theory. When a D7-brane intersects a stack of D7-branes and an O7-plane that forms an SO(10) symmetry, an orientifold mirror D7-brane always intersects the stack of

7-branes at the same intersection curve. Two different kinds of open strings become massless simultaneously on this curve. Codimension-3 singularities along this curve are $b^{(a)}$ that we have already mentioned, and zero locus of $R^{(4)} = a_2^2 - 4a_0a_4$.

The two collapsed two-cycles turns into one another, when they are traced around one of the codimension-3 singularities at a zero of $R^{(4)}$. Therefore, it is convenient to think of a covering curve $\tilde{\bar{c}}_{(\mathbf{vec})}$ that traces the collapsed two-cycles. $\tilde{\bar{c}}_{(\mathbf{vec})}$ is a degree-2 cover of $\bar{c}_{(\mathbf{vec})}$, and ramifies at the zero locus of $R^{(4)}$. Divisor of the branch points on $\bar{c}_{(\mathbf{vec})}$ is denoted by $b^{(c)}$, and that of ramification points on $\tilde{\bar{c}}_{(\mathbf{vec})}$ by $\tilde{b}^{(c)}$. This degree-two cover $\nu_{\bar{c}_{(\mathbf{vec})}} : \tilde{\bar{c}}_{(\mathbf{vec})} \rightarrow \bar{c}_{(\mathbf{vec})}$ has branch cuts whose number is given by the half of (106). The genus of the covering curve is given by (138). Chiral multiplets on this matter curves are

$$H^0\left(\tilde{\bar{c}}_{(\mathbf{vec})}; \mathcal{O}\left(i^* K_\Sigma + \tilde{b}^{(c)}\right) \otimes \mathcal{L}_G \otimes \rho_{(\mathbf{vec})}(\mathcal{E})\right). \tag{325}$$

$\mathcal{L}_G$ is defined by field strength given by (288). Since only one collapsed two-cycle is associated with each point in the covering matter curve, it is well-defined as a line bundle on the covering curve. The covering matter curve is where $M2$-brane propagates, and is more appropriate object in describing this group of matter multiplets than the ordinary matter curve $\bar{c}_{(\mathbf{vec})}$. One could also describe the same chiral multiplets, though, as global holomorphic sections of a rank-2 vector bundles on $\bar{c}_{(\mathbf{vec})}$ obtained by pushing forward the line bundle in (325) by $\nu_{\bar{c}_{(\mathbf{vec})}}$.

The two different matter curves $\bar{c}_{(\mathbf{16})}$ and $\bar{c}_{(\mathbf{vec})}$ intersect at codimension-3 singular loci $b^{(a)}$. There are $(4K_\Sigma + \eta) \cdot (3K_\Sigma + \eta)$ such points. This is where Yukawa couplings

$$\Delta W_{(a)} = \mathbf{16}\ \mathbf{16}\ \mathbf{10} \tag{326}$$

can be generated [11, 10]. Simple algebraic relation among the collapsed two-cycles there—$C^p + C^q = C^{pq}$–allows $M2$-branes to reconnect.

7.3 On an $A_4 = E_4$-Singularity Locus

When all $a_{0,2,3,4,5}$ are allowed to be non-zero, S is an $A_4 = E_4$ singular locus. The discriminant of the elliptic fibration (308) is given locally around Σ by (307). See (146) and (150) for the definitions of $P^{(5)}$ and $R^{(5)}$, respectively. No arguments on chiral multiplets from the bulk of Σ have to be changed from the cases with D_5 or E_6 singularities. A bundle $\mathcal{E}$ may be turned on in the $\mathrm{U}(1)_Y$ direction to break the $\mathrm{SU}(5)_{\mathrm{GUT}}$ symmetry of unified theories. The four-form flux to be used in this case is (283), where ω_Y is a two-form on the A_4 singularity

locus Σ, and $C_{A=1,2,3,4}$ are collapsed four two-cycles forming a basis whose intersection form is $(-C_{A_4})$.

A matter curve $\bar{c}_{(\mathbf{10})}$ is the zero locus of a_5, along which the A_4 singularity on a generic point of Σ is enhanced to D_5. There are two groups of codimension-3 singularities on $\bar{c}_{(\mathbf{10})}$. One is where a_4 also vanishes (type (a) intersection points), and the other is where a_3 does (type (d) intersection points). Those points define divisors $b^{(a)}$ and $b^{(d)}$ on $\bar{c}_{(\mathbf{10})}$. See Figure 4. Chiral multiplets on $\bar{c}_{(\mathbf{10})}$ are

$$H^0\left(\bar{c}_{(\mathbf{10})}; \mathcal{O}\left(i^* K_\Sigma + \frac{1}{2} b^{(a)}\right) \otimes \mathcal{L}_G \otimes \rho_{(\mathbf{10})}(\mathcal{E})\right), \tag{327}$$

$$H^0\left(\bar{c}_{(\mathbf{10})}; \mathcal{O}\left(i^* K_\Sigma + \frac{1}{2} b^{(a)}\right) \otimes \mathcal{L}_G^{-1} \otimes \rho_{(\overline{\mathbf{10}})}(\mathcal{E})\right). \tag{328}$$

The line bundle $\mathcal{L}_G$ is given by (279), and the net chirality by (285).

Another matter curve $\bar{c}_{(\bar{\mathbf{5}})}$ is the zero locus of $P^{(5)}$, along which the singularity is enhanced to A_5. Codimension-3 singularities along $\bar{c}_{\bar{\mathbf{5}}}$ are $b^{(a)}$, $b^{(d)}$, and $b^{(c)} = \mathrm{div} R^{(5)} + b^{(a)}$. The matter curve $\bar{c}_{(\bar{\mathbf{5}})}$ forms a double point singularity at $b^{(d)}$, and it is convenient to discuss its blow up, the covering matter curve $\tilde{\bar{c}}_{(\bar{\mathbf{5}})}$. This covering matter curve is where $M2$-brane propagates, and not the matter curve $\bar{c}_{(\bar{\mathbf{5}})}$, because each point in the covering matter curve is in one-to-one correspondence with the collapsed two-cycle along the matter curve. Chiral multiplets are global holomorphic sections of line bundles on the covering matter curve,

$$H^0\left(\tilde{\bar{c}}_{(\bar{\mathbf{5}})}; \mathcal{O}\left(i^* K_\Sigma + \frac{1}{2} b^{(c)}\right) \otimes \mathcal{L}_G \otimes \rho_{(\bar{\mathbf{5}})}(\mathcal{E})\right), \tag{329}$$

$$H^0\left(\tilde{\bar{c}}_{(\bar{\mathbf{5}})}; \mathcal{O}\left(i^* K_\Sigma + \frac{1}{2} b^{(c)}\right) \otimes \mathcal{L}_G^{-1} \otimes \rho_{(\mathbf{5})}(\mathcal{E})\right). \tag{330}$$

See Figure 4 for the rough sketch of the geometry of the two matter curves and the variety of their intersection points. Table 2 shows sets of geometric data such as genus of the covering matter curve $\tilde{\bar{c}}_{(\mathbf{5}}$ and the number of various types of codimension-3 singularities for a few examples of η.

The topological relation among the collapsing cycles,

$$C^{ab;p} + C^{cd;q} = \epsilon^{abcde} C_{e;}{}^{pq}, \tag{332}$$

allows a reconnection of $M2$-branes wrapped on the relevant two-cycles and Yukawa couplings of the form (up-type like)

$$\Delta W_{(a)} = \mathbf{10}^{ab}\, \mathbf{10}^{cd}\, \mathbf{5}^e \epsilon_{abcde} \tag{333}$$

may be generated. These types of Yukawa couplings are generated at type (a) intersection points of $\bar{c}_{(\mathbf{10})} \cdot \bar{c}_{(\bar{\mathbf{5}})}$, because, all the two-cycles C^p, C^q and C^{pq} mode C_A $(A = 1, 2, 3, 4)$ collapse to zero size there. Singularity is enhanced from A_4 to E_6 at each type (a) intersection point [1, 11].

Another relation

$$C_{a;}{}^{pq} + C_{b;}{}^{rs} + \epsilon_{pqrst} C^{ab;t} = 0 \tag{334}$$

allows a different kind of reconnection of $M2$-branes, and hence Yukawa couplings (down-type like)

$$\Delta W_{(d)} = \bar{\mathbf{5}}_a \, \mathbf{10}^{ab} \, \bar{\mathbf{5}}_b \tag{335}$$

may be generated. This type of reconnection is possible at the type (d) intersection points, because all the two-cycles C^t, C^{pq} and C^{rs} can collapse to zero size simultaneously there. Singularity is enhanced from A_4 to D_6 there. Three branches of matter curves intersect at each type (d) intersection point (see Figure 4). Local geometry around this type (d) intersection point allows a Type IIB interpretation. This is where a stack of five D7-branes, an O7-plane, one D7-brane and its orientifold mirror image intersect simultaneously.

It is interesting that the up-type and down-type Yukawa couplings are associated with different kinds of the intersection points of the two matter curves $\bar{c}_{(\mathbf{10})}$ and $\bar{c}_{(\bar{\mathbf{5}})}$. This is an important observation in an attempt to understand Yukawa couplings of quarks and leptons.

7.4 On an $A_2 + A_1 = E_3$-Singularity Locus

The SU(6) bundle compactification of the Heterotic theory can be used to study various properties of F-theory vacua with a locus of $A_2 + A_1 = E_3$ singularity. There are three matter curves on the E_3 singularity locus Σ; $\bar{c}_{(Q)}$, where the singularity is enhanced to $A_4 = E_4$, $\bar{c}_{(\bar{U})}$ where the symmetry is enhanced to SU(4) $\times$ SU(2), and finally $\bar{c}_{(L)}$, where the enhanced symmetry is SU(3) $\times$ SU(3). See Figure 6 for how those curves intersect one another. Table 3 shows numerical data of the geometry of those curves for a few examples. Although the analysis in section 5.5 relies on field theory approximation of the Heterotic string theory, qualitative nature of the intersection of those curves are believed to be the same in dual F-theory vacua.

In the Heterotic theory language, the matter curve $\bar{c}_{(Q)}$ is given by $a_6 = 0$. Chiral multiplets Q and Q^c in the $(\bar{\mathbf{3}}, \mathbf{2})$ and $(\mathbf{3}, \mathbf{2})$ representation of the SU(3) $\times$ SU(2) unbroken symmetry group are localized on this curve, and they are identified with the independent

generators of the cohomology groups,

$$H^0\left(\bar{c}_{(Q)}; \mathcal{O}\left(i^* K_\Sigma + \frac{1}{2} b^{(a)}\right) \otimes \mathcal{L}_G\right), \tag{336}$$

$$H^0\left(\bar{c}_{(Q)}; \mathcal{O}\left(i^* K_\Sigma + \frac{1}{2} b^{(a)}\right) \otimes \mathcal{L}_G^{-1}\right); \tag{337}$$

the divisor $b^{(a)}$ is defined by (302) with $N = 6$, and corresponds to the type (a) intersection points in Figure 6.

The matter curve $\bar{c}_{(\bar{U})}$ is defined by (161). Chiral multiplets $\bar{U}$ and $\bar{U}^c$ in the $(\mathbf{3}, \mathbf{1})$ and $(\bar{\mathbf{3}}, \mathbf{1})$ representations correspond to

$$H^0\left(\tilde{\bar{c}}_{(\bar{U})}; \mathcal{O}\left(i^* K_\Sigma + \frac{1}{2} \tilde{b}^{(c)}\right) \otimes \mathcal{L}_G\right), \tag{338}$$

$$H^0\left(\tilde{\bar{c}}_{(\bar{U})}; \mathcal{O}\left(i^* K_\Sigma + \frac{1}{2} \tilde{b}^{(c)}\right) \otimes \mathcal{L}_G^{-1}\right); \tag{339}$$

here, the covering matter curve $\tilde{\bar{c}}_{(\bar{U})}$ is obtained by blowing up and resolving triple points of the curve $\bar{c}_{(\bar{U})}$, the type (e) points in Figure 6. The divisor $b^{(c)}$ is given by $\mathrm{div} R^{(6)} + 2b^{(a)}$.

The last group of chiral multiplets, denoted by L, come from

$$H^0\left(\bar{c}_{(L)}; \mathcal{O}\left(i^* K_\Sigma + \frac{1}{2} b^{(f)}\right) \otimes \mathcal{L}_G\right) \oplus H^0\left(\bar{c}_{(L)}; \mathcal{O}\left(i^* K_\Sigma + \frac{1}{2} b^{(f)}\right) \otimes \mathcal{L}_G^{-1}\right). \tag{340}$$

These matter multiplets are localized on the curve $\bar{c}_{(L)}$. Its defining equation is given by the zero locus of (184). The divisor $b^{(f)}$ is the zero locus of (201).

The duality map (240–251) between the moduli parameters of the Heterotic and F-theories was established only for bundles with a rank $N \leq 5$. Thus, the divisors $b^{(c)}$ and $b^{(f)}$ are still characterized in terms of the data $a_{0,2,3,4,5,6}$ describing the vector bundle V. Thus, we have not seen for the rank-6 bundle compactification of F-theory that those divisors correspond to codimension-3 singularities in F-theory.

There are three types of the way matter curves intersect, as we see in Figure 6. Topological relations of collapsed two-cycles are all different for those different kinds of intersection points. Thus, the interactions generated at the codimension-3 singularities are different for different types of singularities. At type (a) intersection points,

$$\Delta W_{(a)} = Q^{a\alpha} Q^{b\beta} \bar{U}^c_{ab} \epsilon_{\alpha\beta} \tag{341}$$

may be generated, where a, b are SU(3) indices and α, β SU(2) indices. At the type (e) points, we may have

$$\Delta W_{(e)} = \bar{U}_a \bar{U}_b \bar{U}_c \epsilon^{abc}. \tag{342}$$

The other type of three point couplings is

$$\Delta W_{(d)} = Q^{a\alpha}\bar{U}_a L_\alpha, \tag{343}$$

and this type of interactions may be generated at the type (d) points. The enhanced singularity at the type (d) points is A_5, and this interaction is what we expect when six D7-branes are separated into three, two and one coincident D7-branes intersecting one another [47].

Suppose that the low-energy spectrum of chiral multiplets consists of the minimal anomaly free choice. That is, the cohomology groups (336), (338) and (340) have one, two and one independent generators, respectively, and all other cohomology groups vanish. Then, the type (e) Yukawa couplings vanish because of the anti-symmetric nature of the contraction of the SU(3) indices, and the type (a) Yukawa interactions simply do not exist because there is no particle like $\bar{U}^c$ at low-energy. Thus, the effective theory is only with the type (d) Yukawa interaction. This model is known as the 3–2 model, one of the most famous calculable models of dynamical supersymmetry breaking [50].

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A Direct Image as a Pushforward from its Support

If a sheaf $\mathcal{E}$ on an algebraic variety X is supported on a closed subvariety $i: Y \hookrightarrow X$, there exists a sheaf of Abelian group $\mathcal{F}$ on Y such that $\mathcal{E} = i_*\mathcal{F}$ as a sheaf of Abelian group. It is not true in general, however, that there exists a sheaf of $\mathcal{O}_Y$-module $\mathcal{F}$ such that $\mathcal{E} = i_*\mathcal{F}$ as a sheaf of $\mathcal{O}_X$-module.

Any locally holomorphic functions on X acts on a sheaf of the form $i_*\mathcal{F}$ by restricting them on Y first, and then by multiplying them to $\mathcal{F}$. Thus, in order to see whether $\mathcal{E}$ on X is given by a pushforward of a sheaf of $\mathcal{O}_Y$-module $\mathcal{F}$, one needs to make sure that any local

sections of ideal sheaf of Y, $\mathcal{I}_Y$ in

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0, \tag{344}$$

acts trivially on $\mathcal{E}$.

To get a feeling of when Fourier–Mukai transforms (19) of bundles $\rho(V)$ and direct images $R^1\pi_{Z*}\rho(V)$ are given by pushforward of sheaves of modules, we explicitly calculate direct images for a simple case using Čech cohomology.

A.1 Warming Up

We would like to calculate direct images associated with elliptic fibration $\pi_Z : Z \to B_2$. Before discussing $R^i\pi_{Z*}\rho(V)$, however, we begin with an elementary exercises of calculating sheaf cohomology on an elliptic curve E.

Suppose that an elliptic curve E is given by an equation

$$ZY^2 = X^3 + f_0 XZ^2 + g_0 Z^3 \tag{345}$$

in $\mathbb{P}^2$, where $[X : Y : Z]$ are homogeneous coordinates of $\mathbb{P}^2$. E is covered by two Affine open sets, U_Z and U_Y. U_Z is E in $\mathbb{P}^2 \backslash \{Z = 0\}$, and hence is $E\backslash\{e_0 = \infty\}$. In U_Z, the defining equation of E above can be written in terms of Affine coordinates $(x, y) = (X/Z, Y/Z)$. U_Y is obtained by removing three points in E specified by intersection of E and a hyperplane $Y = 0$. This choice of hyperplane $Y = 0$ is rather arbitrary; it could have been any other hyperplane, as long as it is not $Z = 0$. When the $Y = 0$ hyperplane is used, then the three points p_i $(i = 1, 2, 3)$ that are not contained in E correspond to $(x, y) = (x_i, 0)$, with x_i's three roots of $x^3 + f_0 x + g_0 = 0$. Affine open sets U_Z and U_Y cover the entire E, and this open covering can be used to calculate Čech cohomology [27].

Let us begin with calculation of $H^i(E; \mathcal{O}_E)$ $(i = 0, 1)$ in terms of Čech cohomology. In the Čech complex

$$0 \to C^0 \to C^1 \to 0, \tag{346}$$

$$\begin{aligned} C^0 &= \{\varphi_Z \in \mathbb{C}(E) | \varphi \in \mathbb{C}[E\backslash e_0]\} \oplus \{\varphi_Y \in \mathbb{C}(E) | \varphi \in \mathbb{C}[E\backslash\{p_i\}_{i=1,2,3}]\}, && (347) \\ C^1 &= \{\varphi \in \mathbb{C}(E) | \varphi \in \mathbb{C}[E\backslash\{e_0, p_1, p_2, p_3\}]\}. && (348) \end{aligned}$$

Here, $\mathbb{C}(E)$ means a set of rational functions on E, and $\mathbb{C}[U]$ a set of regular functions on $U \subset E$. Now, $H^0(E; \mathcal{O}_E)$ is a subset of C^0 given by $\{(\varphi_Z, \varphi_Y) | \varphi_Z = \varphi_Y \in U_Z \cap U_Y\}$. Thus,

φ_Y cannot have a pole at $p_{1,2,3}$ because φ_Z does not, and φ_Z cannot have a pole at e_0 because φ_Y does not. Thus, $\varphi_Z = \varphi_Y$ should be chosen as a function that is regular everywhere in E, which means that they are a constant-valued function. This is how one can obtain a well-known result $H^0(E;\mathcal{O}_E) \simeq \mathbb{C}$ in Čech cohomology.

$H^1(E;\mathcal{O}_E)$ is generated by either one of functions

$$\frac{y}{x-x_i} \qquad (i=1,2,3). \tag{349}$$

These functions are regular on $U_Z \cap U_Y$, and have poles of order one at e_0 and p_i, but nowhere else. They are elements of C^1. If the functions (349) on $U_Z \cap U_Y$ are to be expressed as $\varphi_1 - \varphi_2$ with $\varphi_1 \in C^0_Z$ and $\varphi_2 \in C^0_Y$, the order-one pole at e_0 should come from φ_1 and the order-one pole at p_i from φ_2. Both φ_1 and φ_2 have to be regular otherwise. Since any elliptic functions have at least a pole of order two or two poles of order one, no elliptic functions have properties required for φ_1 or φ_2. Thus, the functions (349) belongs to the cokernel of $C^0 \to C^1$, and hence can be a generator of $H^1(E;\mathcal{O}_E)$. On the other hand, difference between any two functions in (349) does not have a pole at e_0, and hence it can be expressed as $\varphi_1 - \varphi_2$. Thus, there is only one independent generator out of three functions in (349). This is how one can understand $h^1(E;\mathcal{O}_E) = 1$.

We will work on another exercise: $H^i(E;\mathcal{O}(p-e_0))$ $(i=0,1)$. It is well known that both cohomology groups vanish for $p \neq e_0$, but it is an instructive exercise to reproduce this result, before taking on an even more difficult problem, calculation of $R^1\pi_*$ in elliptic fibration given by π. For $\mathcal{O}_E(p-e_0)$,

$$\begin{aligned}
C^0 &= C^0_Z \oplus C^0_Y, & (350)\\
C^0_Z &= \{\varphi_Z \in \mathbb{C}(E) | \varphi_Z \in \mathbb{C}[E\backslash\{e_0,p\}],\ v_p(\varphi_Z) \geq -1\}, & (351)\\
C^0_Y &= \{\varphi_Y \in \mathbb{C}(E) | \varphi_Y \in \mathbb{C}[E\backslash\{p,p_{1,2,3}\}],\ v_p(\varphi_Y) \geq -1,\ v_{e_0}(\varphi_Y) \geq 1\}, & (352)\\
C^1 &= \{\varphi \in \mathbb{C}(E) | \varphi \in \mathbb{C}[E\backslash\{e_0,p,p_{1,2,3}\}],\ v_p(\varphi) \geq -1\}. & (353)
\end{aligned}$$

Here, $v_p(\varphi) = m$ means that a rational function φ has a zero of order m at a point p if $m > 0$, and φ has a pole of order $(-m)$ at a point p if $m < 0$. Thus, global holomorphic section, (φ_Z, φ_Y) such that $\varphi_Z - \varphi_Y = 0$ on $U_Z \cap U_Y$, has to be constant-valued function with $\varphi_Y(e_0) = 0$. Thus, there is only one trivial holomorphic section, $\varphi_Z = \varphi_Y = 0$, and $h^0(E;\mathcal{O}_E(p-e_0)) = 0$.

Let us now turn to $H^1(E;\mathcal{O}_E(p-e_0))$. Although the functions (349) generate the coho-

mology group $H^1(E;\mathcal{O}_E)$, they are now decomposed into

$$\frac{y}{x-x_i} = \frac{y+y(p)}{x-x(p)} - \frac{y(x(p)-x_i)+y(p)(x-x_i)}{(x-x(p))(x-x_i)} =: \varphi_1 - \varphi_2. \tag{354}$$

Here, $(x(p),y(p))$ are the coordinates of the point p. φ_1 has poles of order one at p and e_0, nowhere else. Thus, $\varphi_1 \in C^0_Z$. φ_2 has poles of order one at p and p_i, but e_0 is a zero of order one for φ_2. Thus, $\varphi_2 \in C^0_Y$. Thus, the functions (349) do not generate the cokernel of $C^0 \to C^1$, and hence the cohomology group $H^1(E;\mathcal{O}_E(p-e_0))$ is trivial. Note also that the decomposition $\varphi_1 \in C^0_Z$ and $\varphi_2 \in C^0_Y$ is actually unique.

A.2 $R^1\pi_*\mathcal{O}(C-\sigma)$

Fourier–Mukai transforms

$$R^1 p_{1*}\left[p_2^*(\rho(V)) \otimes \mathcal{P}_B^{-1} \otimes \mathcal{O}(-q^* K_{B_2})\right] \tag{355}$$

for an elliptic fibration $p_1 : Z \times_B Z \to Z$ and direct images

$$R^1 \pi_{Z*}\rho(V) \tag{356}$$

for an elliptic fibration $\pi_Z : Z \to B_2$ often vanishes apart from closed subsets $C_{\rho(V)} \hookrightarrow Z$ and $\bar{c}_{\rho(V)} \hookrightarrow B_2$. We address a question here, whether those sheaves with their support on closed subvarieties are expressed as pushforwards of sheaves of $\mathcal{O}_{C_{\rho(V)}}$-modules and $\mathcal{O}_{\bar{c}_{\rho(V)}}$-modules, respectively.

We restrict our attention to cases where the support subvariety codimension-1 has its well-defined normal coordinate. Since the question—whether the direct images are expressed as pushforwards or not—is about a local property, the following argument is valid wherever a normal coordinate to the support is well defined locally. If the normal coordinate is well defined, then the question is answered by checking whether multiplication of the normal coordinate (seen as a function) upon the generators of direct images is trivial or not.

For a bundle V given by spectral cover construction, V is locally expressed as $\oplus_i(p_i - e_0)$, with p_i varying over the coordinates of base manifold. $\rho(V)$ also share the same property. Multiplication of $\mathcal{P}_B^{-1}$ in Fourier–Mukai transform does not change this structure, either. Thus, it is sufficient to deal with individual summand, all of which are of the form $\mathcal{O}(C-\sigma)$ with C describing a locus of p_i, varying over the base manifold. Because of the nature of our question we try to address, it is sufficient to maintain only the normal direction in the

base manifold; a normal direction of $C_{\rho(V)}$ in Z, or that of $\bar{c}_{\rho(V)}$ in B_2. We will use t as the coordinate of this transverse direction.

$R^1\pi_*\mathcal{O}(C-\sigma)$ is trivial around a point away from $C_{\rho(V)}$ or $\bar{c}_{\rho(V)}$. We only need to allow the coefficients in C^0 and C^1 in (350–353) to be holomorphic functions of t. Arguments there leading to $h^1(E;\mathcal{O}(p-e_0))=0$ does not need to be changed.

On the support locus of $R^1\pi_*\mathcal{O}(C-\sigma)$, $C\cdot\sigma$, things are different. Let us take an open set U on the base manifold that contains some section of $C\cdot\sigma$. Then,

$$[R^1\pi_*\mathcal{O}(C-\sigma)](U) = H^1(\pi^{-1}(U);\mathcal{O}(C-\sigma)|_{\pi^{-1}(U)}). \tag{357}$$

$\pi^{-1}(U)$ is covered by two open sets, U_Z and U_Y; notion of $Z=0$ and $Y=0$ are well-defined in elliptic fibration. C^0_Z, for example, is given by

$$C^0_Z = \mathrm{Span}_{\mathbb{C}}\left\{1,x,y,x^2,\cdots\right\}\otimes\mathbb{C}[t] + \varphi_1\otimes(t\mathbb{C}[t]). \tag{358}$$

φ_1 is not contained in C^0_Z, because, as we see shortly, $v_{t=0}(\varphi_1)=-1$. This is why only $t\times\mathbb{C}[t]$ are allowed as a coefficient of φ_1. Similarly, φ_2 does not belong to C^0_Y without being multiplied by the transverse coordinate t. This means that

$$\frac{y}{x-x_i} = \varphi_1 - \varphi_2 \tag{359}$$

in C^1 is not expressed as an image from C^0; the decomposition into φ_1 and φ_2 was unique, but neither φ_1 nor φ_2 belong to C^0_Z or C^0_Y. Thus, these functions ($i=1,2,3$) generate the cohomology group $H^1(\pi^{-1}(U);\mathcal{O}(C-\sigma))$. On the other hand, once it is multiplied by the transverse coordinate t,

$$t\times\frac{y}{x-x_i} = t\times\varphi_1 - t\times\varphi_2 \tag{360}$$

is in the image from C^0. Therefore, i) there is a non-vanishing direct image $R^1\pi_*\mathcal{O}(C-\sigma)$ that is localized on $C\cdot\sigma$, and ii) multiplication of the transverse coordinate t annihilates it. Thus, the ideal sheaf of $C\cdot\sigma$ acts trivially on $R^1\pi_*\mathcal{O}(C-\sigma)$, and it is given by a pushforward of a sheaf on $C\cdot\sigma$ as a sheaf of $\mathcal{O}_{B_2}$-module. Furthermore, iii) the sheaf on $C\cdot\sigma$ is rank-1, (the same argument after (349) is applied also here) and iv) since generator $y/(x-x_i)$ transforms like a section of $\mathcal{L}_H^{\otimes 3}\otimes\mathcal{L}_H^{\otimes -2}\simeq\mathcal{L}_H$, its coefficient function transforms as $\mathcal{L}_H^{-1}\simeq\mathcal{O}(K_{B_2})$.

We have yet to verify

$$v_{t=0}(\varphi_1) = v_{t=0}(\varphi_2) = -1. \tag{361}$$

As a point p approaches to e_0, $(x(p),y(p))$ goes to (∞,∞). To see geometry of elliptic curve around the infinity point, it is better to use $(\xi,\zeta):=(X/Y,Z/Y)$ as the Affine coordinate

system. The defining equation of elliptic curve becomes

$$\zeta = \xi^3 + f_0 \xi \zeta^2 + g_0 \zeta^3, \tag{362}$$

and the infinity point e_0 corresponds to $(\xi, \zeta) = (0, 0)$. ξ can be chosen as a local coordinate on E around the infinity point e_0, and $\zeta \simeq \xi^3$ approximately. Thus, as a point p approaches e_0,

$$x(p) = \left(\frac{\xi}{\zeta}\right)(p) \simeq \frac{1}{\xi(p)^2}, \tag{363}$$

$$y(p) = \frac{1}{\zeta(p)} \simeq \frac{1}{\xi(p)^3}. \tag{364}$$

By extracting the leading behavior of $\xi(p) \to 0$, we find that

$$\varphi_1 \simeq -\frac{y(p)}{x(p)} \simeq -\frac{1}{\xi(p)}, \tag{365}$$

$$\varphi_2 \simeq \frac{y(p)(x - x_i)}{x(p)(x - x_i)} \simeq -\frac{1}{\xi(p)}. \tag{366}$$

Since we chose t as the transverse coordinate to $C_{\rho(V)}$ or $\bar{c}_{\rho(V)}$, $\xi(p) \propto t$ around $C \cdot \sigma$. Thus, we verified (361).

B Appendices to Section 5

B.1 A Relation between $\chi(V)$ and $\chi(\wedge^2 V)$

In this section, prove (375), which gives a relation between $\chi(V)$ and $\chi(\wedge^2 V)$. This relation can be used as a consistency check of the computation, or as a shortcut to obtain $\chi(\wedge^2 V)$ from $\chi(V)$.

For a $\mathrm{U}(N)$ bundle V, one can show that

$$\mathrm{c}_1(\wedge^2 V) = (N-1) c_1(V), \tag{367}$$

$$\mathrm{ch}_3(V) = \frac{1}{2} c_3(V) - \frac{1}{2} c_2(V) c_1(V) + \frac{1}{6} c_1(V)^3, \tag{368}$$

$$\mathrm{ch}_3(\wedge^2 V) = (N-4)\mathrm{ch}_3(V) - c_2(V) c_1(V) + \frac{1}{2} c_1(V)^3. \tag{369}$$

See e.g. an appendix B of [18].

For a Calabi–Yau 3-fold Z ($c_1(TZ) = 0$),

$$\chi(Z;V) = \int_Z \mathrm{ch}_3(V) + \int_Z \frac{c_2(TZ)}{12} c_1(V), \tag{370}$$

$$\begin{aligned}
\chi(Z;\wedge^2 V) &= \int_Z \mathrm{ch}_3(\wedge^2 V) + \int_Z \frac{c_2(TZ)}{12} c_1(\wedge^2 V) \\
&= (N-4)\int_Z \mathrm{ch}_3(V) + (N-1)\int_Z \frac{c_2(TZ)}{12} c_1(V) \\
&\qquad - \int_Z c_2(V) c_1(V) + \frac{1}{2}\int_Z c_1(V)^3.
\end{aligned} \tag{371}$$

Since for elliptic fibration $\pi_Z : Z \to B$ we have

$$c_2(TZ) = \sigma \cdot 12 c_1(TB) + \cdots, \tag{372}$$

$$c_2(V) = \sigma \cdot \eta + \cdots, \tag{373}$$

$$c_1(V)^3 = 0 \qquad \text{for } c_1(V) = \pi_Z^* \pi_{C*} \gamma, \tag{374}$$

we finally obtain

$$\chi(\wedge^2 V) = (N-4)\chi(V) + (3K_{B_2} + \eta) \cdot c_1(V), \tag{375}$$

where $\chi(V) := -\chi(Z;V)$ and and $\chi(\wedge^2 V) := -\chi(Z;\wedge^2 V)$.

B.2 Calculation of $\deg r|_D$ and $\deg j^* r$

We present an explicit calculation of $\deg r|_D$ on D from individual type (a) intersection points. In other words, we calculate the multiplicity of intersection of two curves r and D in the spectral surface C_V. The type (a) intersection points of r and D are also where the ramification divisor r intersects the matter curve $\bar{c}_V = \sigma|_{C_V}$. We will also see below through explicit calculation that the multiplicity of the intersection of r and $\bar{c}_V$ is 1, although that is already clear from an argument presented in the text.

In order to find out the multiplicity of intersection of two curves on a surface, only local geometry of the surface matters. We will first describe local geometry of the spectral surface C_V around a type (a) intersection point, and find the defining equations of the curves r, D and $\bar{c}_V$. It is quite easy, then, to find out the multiplicity of intersection.

B.2.1 For a rank $V = 3$ Case

Because the type (a) intersection points are always on the zero section σ, it is convenient to use the coordinates (ξ, ζ) in describing the direction of the elliptic fiber. Furthermore, since we focus on a local geometry of C_V, we can use $\zeta \sim \xi^3$.

Type (a) intersection points are found wherever both a_N and a_{N-1} vanish on the base 2-fold B_2. Here, $N := \text{rank } V$. We choose a local patch of one of such points in B_2, and set coordinates in the patch so that $a_N = u$, and $a_{N-1} = -v$. In a local patch of Z with a set of coordinates (u, v, ξ), the spectral surface of a rank-3 bundle V is given by

$$u - v\xi + \xi^3 = 0, \tag{376}$$

where we have set $a_0 = 1$ at the point; we do not lose generality by doing so, because the coordinates u and v can be rescaled if necessary. We study the rank $V = 3$ case first. This defining equation was used when drawing the spectral surface C_V in Figure 1.

An appropriate choice of local coordinates on C_V is (ξ, v), whereas (u, v) can be used for the base 2-fold B_2. The projection $\pi_C : C_V \to B_2$ is given by

$$\begin{array}{rlcl} \pi_C : & p & \mapsto & b = \pi_C(p), \\ & (\xi, v) & \mapsto & (u, v) = (v\xi - \xi^3, v). \end{array} \tag{377}$$

Thus,

$$\pi_C^*(du \wedge dv) = (v - 3\xi^2) d\xi \wedge dv, \tag{378}$$

and hence

$$r = \text{div } (v - 3\xi^2). \tag{379}$$

Note also that

$$\pi_C^*(u) = \xi(v - \xi^2), \quad \text{and} \quad \text{div } \pi_C^*(u) = \text{div } \xi + \text{div } (v - \xi^2) = \bar{c}_V + D. \tag{380}$$

Thus, r and D intersects with multiplicity 2, while r and $\bar{c}_V$ with multiplicity 1. See Figure 8.

B.2.2 For rank $V \geq 4$ Cases

For bundles with rank $V \geq 4$, local defining equation of the spectral surface becomes

$$u - v\xi + \xi^2 = 0, \tag{381}$$

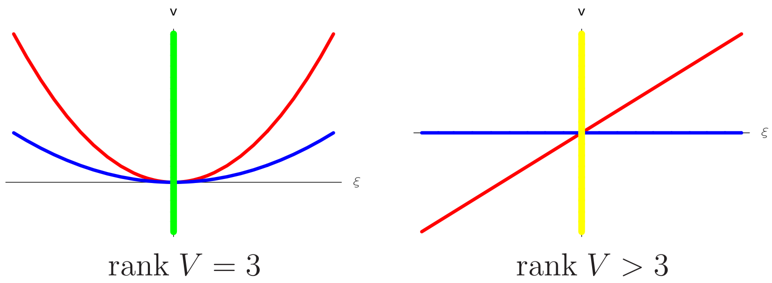

Figure 8: This figure shows how the curves r, D and $\bar{c}_V$ intersect on C_V at a type (a) point.

where now local coordinates of B_2 around a type (a) point are chosen so that $a_N \sim u$ and $a_{N-1} \sim -v$, and we set $(a_{N-2}x^{(N-2)/2} + \cdots + a_0)\xi^{N-2} = 1$ at a type (a) intersection point without a loss of generality. The difference from (376) is due to the fact that the a_{N-2} term is absent in the equation determining the spectral surface of rank-3 bundles. The left panel of Figure 2 was drawn using the defining equation above.

We can use (ξ, v) as the local coordinates on C_V, and the projection $\pi_C : C_V \to B_2$ is given by

$$\begin{array}{rlcl} \pi_C : & p & \mapsto & b = \pi_C(b), \\ & (\xi, v) & \mapsto & (u, v) = (v\xi - \xi^2, v) \end{array} \tag{382}$$

and the ramification divisor can be read out from

$$\pi_C^*(du \wedge dv) = (v - 2\xi) d\xi \wedge dv; \tag{383}$$

now we have $r = \text{div } (v - 2\xi)$. Note also that $D = \text{div } v$ and $\bar{c}_V = \text{div } \xi$. Thus, the curves r and D intersect with multiplicity 1 at a type (a) intersection point for bundles with rank $V > 3$. The two curves $\bar{c}_V$ and r also intersect transversely.

B.3 Geometry of $C_{\wedge^2 V}$ around the Pinch Points

In this appendix, geometry of the spectral surface $C_{\wedge^2 V}$ for rank-4 bundles V is discussed.

For any point $b \in B_2$, the spectral surface C_V for the bundle in the fundamental representation determines four points $\{p_i, p_j, p_k, p_l\}$ in the elliptic fiber of b, E_b. Points on B_2 satisfying both $a_3 = 0$ and $R^{(4)} := a_2^2 - 4a_4a_0 = 0$ are called type (c) points. We will focus on a local neighborhood $U \subset B_2$ of a type (c) point, and determine the behavior of $C_{\wedge^2 V}$ in $\pi_Z^{-1}(U) \subset Z$.

We can choose $\tilde{a}_3 \equiv a_3/a_4$ and $\tilde{R}^{(4)} \equiv R^{(4)}/a_4^2 = (a_2^2 - 4a_4 a_0)/a_4^2$ as a set of local coordinates. Then, the coordinates of the four points of C_V in the fiber direction are determined as functions of the coordinates of the base manifold for small $(\tilde{a}_3, \tilde{R}^{(4)})$:

$$p_i : \qquad (x,y) \sim \left(x_* + \frac{1}{2}\sqrt{\tilde{R}^{(4)} - 4y_*\tilde{a}_3}, \; + \left(y_* + \frac{3x_*^2 + f_0}{4y_*}\sqrt{\tilde{R}^{(4)} - 4y_*\tilde{a}_3} \right) \right), \tag{384}$$

$$p_j : \qquad (x,y) \sim \left(x_* + \frac{1}{2}\sqrt{\tilde{R}^{(4)} + 4y_*\tilde{a}_3}, \; - \left(y_* + \frac{3x_*^2 + f_0}{4y_*}\sqrt{\tilde{R}^{(4)} + 4y_*\tilde{a}_3} \right) \right), \tag{385}$$

$$p_k : \qquad (x,y) \sim \left(x_* - \frac{1}{2}\sqrt{\tilde{R}^{(4)} - 4y_*\tilde{a}_3}, \; + \left(y_* - \frac{3x_*^2 + f_0}{4y_*}\sqrt{\tilde{R}^{(4)} - 4y_*\tilde{a}_3} \right) \right), \tag{386}$$

$$p_l : \qquad (x,y) \sim \left(x_* - \frac{1}{2}\sqrt{\tilde{R}^{(4)} + 4y_*\tilde{a}_3}, \; - \left(y_* - \frac{3x_*^2 + f_0}{4y_*}\sqrt{\tilde{R}^{(4)} + 4y_*\tilde{a}_3} \right) \right), \tag{387}$$

where $p_i = p_k = (x_*, +y_*)$ and $p_j = p_l = (x_*, -y_*)$ are the four points right on the type (c) point, and $x_* = -a_2/(2a_4)$ and $(y_*)^2 = x_*^3 + f_0 x_* + g_0$. Only the leading order deviation from $(x_*, \pm y_*)$ for small $(\tilde{a}_3, \tilde{R}^{(4)})$ are maintained in the expressions above, and higher order dependence on $(\tilde{a}_3, \tilde{R}^{(4)})$ is dropped.

Now that the coordinates in the fiber direction are given for the four points of C_V in each fiber, one can determine the coordinates for the six points of $C_{\wedge^2 V}$ in each fiber. If two points p_1 and p_2 on an elliptic curve are given coordinate values $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$, then the coordinates of their group-law sum $p_1 \boxplus p_2 = (x_{1\boxplus 2}, y_{1\boxplus 2})$ are given by

$$x_{1\boxplus 2} = -x_1 - x_2 + \left(\frac{y_1 - y_2}{x_1 - x_2} \right)^2, \tag{388}$$

$$y_{1\boxplus 2} = -\frac{(y_1 - y_2)x_{1\boxplus 2} + (x_1 y_2 - x_2 y_1)}{(x_1 - x_2)}. \tag{389}$$

Using this addition theorem, one can calculate (x, y) for all six $p_i \boxplus p_j$ $(1 \leq i < j \leq 4)$.

We know that two points $p_i \boxplus p_j$ and $p_k \boxplus p_l$ are on the zero section on the matter curve $\bar{c}_{\wedge^2 V}$ specified by $\tilde{a}_3 \propto a_3 = 0$. We also know that $p_k = p_i$ and $p_l = p_j$ on the type (c) points, and hence $p_i \boxplus p_l$ and $p_j \boxplus p_k$ are also on the zero section. We are interested in how those four points on $C_{\wedge^2 V}$ behave around the zero section in $\pi_Z^{-1}(U)$. For the purpose of describing geometry around the zero section, $\xi \sim x/y$ is more useful coordinate in the fiber direction

than the pair (x, y). After a little calculation, one finds that

$$\xi(p_i \boxplus p_j) \sim \frac{-\sqrt{\tilde{R}^{(4)} - 4y_*\tilde{a}_3} + \sqrt{\tilde{R}^{(4)} + 4y_*\tilde{a}_3}}{4y_*}, \tag{390}$$

$$\xi(p_k \boxplus p_l) \sim \frac{+\sqrt{\tilde{R}^{(4)} - 4y_*\tilde{a}_3} - \sqrt{\tilde{R}^{(4)} + 4y_*\tilde{a}_3}}{4y_*}, \tag{391}$$

$$\xi(p_i \boxplus p_l) \sim \frac{-\sqrt{\tilde{R}^{(4)} - 4y_*\tilde{a}_3} - \sqrt{\tilde{R}^{(4)} + 4y_*\tilde{a}_3}}{4y_*}, \tag{392}$$

$$\xi(p_j \boxplus p_k) \sim \frac{+\sqrt{\tilde{R}^{(4)} - 4y_*\tilde{a}_3} + \sqrt{\tilde{R}^{(4)} + 4y_*\tilde{a}_3}}{4y_*}, \tag{393}$$

The geometry (i.e., $C_{\wedge^2 V}$) that those four points sweep is better parametrized by

$$w_\pm := \sqrt{\tilde{R}^{(4)} \mp 4y_*\tilde{a}_3} \tag{394}$$

With these two parameters, the spectral surface $C_{\wedge^2 V}$ is locally described by

$$\left(\xi, \tilde{a}_3, \tilde{R}^{(4)}\right) \sim \left(\frac{w_+ + w_-}{4y_*}, -\frac{w_+^2 - w_-^2}{8y_*}, \frac{w_+^2 + w_-^2}{2}\right). \tag{395}$$

The two panels in Figure 2 showing $C_{\wedge^2 V}$ were obtained in this way. The defining equation of $C_{\wedge^2 V}$ in the ambient space Z with coordinates $(\xi, \tilde{a}_3, \tilde{R}^{(4)})$ is obtained by erasing w_+ and w_-. It is

$$\tilde{a}_3^2 = \xi^2 \tilde{R}^{(4)} - 4y_*^2 \xi^4. \tag{396}$$

From this equation, one can see that $(\xi, \tilde{a}_3) = (0, 0)$ is a double point in the $(\xi, \tilde{a}_3)$ plane for ${}^\forall \tilde{R}^{(4)} \neq 0$, and hence the $(\xi, \tilde{a}_3) = (0, 0)$ locus is a double curve. Furthermore, at the type (c) point, $\tilde{R}^{(4)} = 0$, $C_{\wedge^2 V}$ becomes more singular; it is called a pinch point.

On a generic point on the matter curve $\bar{c}_{\wedge^2 V}$, two branches of $C_{\wedge^2 V}$, $p_i \boxplus p_j$ and $p_k \boxplus p_l$ approach the zero section as $\xi \sim \pm \tilde{a}_3/\sqrt{\tilde{R}^{(4)}} \propto \tilde{a}_3$, where $\tilde{a}_3$ is the coordinate transverse to the matter curve. At a type (c) point, on the other hand, two points $p_i \boxplus p_j$ and $p_k \boxplus p_l$ approach as $\xi \propto \mp(w_+ - w_-)$, and two others $p_i \boxplus p_l$ and $p_k \boxplus p_j$ as $\xi \propto \mp(w_+ + w_-)$. The transverse coordinate $\tilde{a}_3 \propto (w_+ + w_-)(w_+ - w_-)$ contains both factors. It is necessary to know these behavior, when one tries to determine how the sheaf of $\mathcal{O}_{B_2}$-module $R^1\pi_{Z*} \wedge^2 V$ responds to the action of the ideal sheaf of the matter curve $\bar{c}_{\wedge^2 V}$.

Finally, let us study the geometry of $\widetilde{C}_{\wedge^2 V}$ introduced in section 4. $\widetilde{C}_{\wedge^2 V}$ is obtained by resolving two branches of $C_{\wedge^2 V}$ along the double-curve singularity into two disjoint components. Because we did not specify in section 4 how to define $\widetilde{C}_{\wedge^2 V}$ around the pinch points of $C_{\wedge^2 V}$, we will obtain $\widetilde{C}_{\wedge^2 V}$ along the double-curve singularity, and extrapolate it to the pinch points to see what happens there.

$\widetilde{C}_{\wedge^2 V}$ is obtained as a strict transform of $C_{\wedge^2 V}$, when the ambient space Z is blown up with a center along the double curve singularity of $C_{\wedge^2 V}$. Let $\tilde{Z}$ be the blow up of Z. Since the double-curve locus is $(\xi, \tilde{a}_3) = (0,0)$, we consider a blow up of Z centered at $(\xi, \tilde{a}_3) = (0,0)$. The coordinate system $(\xi, u, v) \equiv (\xi, \tilde{a}_3, \tilde{R}^{(4)})$ of a local neighborhood $\pi^{-1}(U) \subset Z$ is replaced by those of two patches of $\nu_Z^{-1}(\pi_Z^{-1}(U)) \subset \tilde{Z}$, $(\xi, \tilde{u}, v)$ in one patch and $(\tilde{\xi}, u, v)$ in the other. The two patches are glued together under an identification

$$u = \xi \tilde{u}, \qquad \xi = u \tilde{\xi}. \tag{397}$$

These relations also determine the map $\nu_Z : \tilde{Z} \to Z$.

In the first patch of $\tilde{Z}$, the defining equation of $\widetilde{C}_{\wedge^2 V}$ becomes

$$\tilde{u}^2 = v - 4 y_*^2 \xi^2. \tag{398}$$

Although this blow-up was intended to resolve the double-curve singularity of $C_{\wedge^2 V}$ at $(\xi, u) = (0,0)$ and $v \neq 0$, this process also resolves the codimension-2 singularity at $(\xi, u, v) = (0,0,0)$; $\widetilde{C}_{\wedge^2 V}$ given by the equation above is smooth even at $(\xi, \tilde{u}, v) = (0,0,0)$. We use the equation above as the definition of $\widetilde{C}_{\wedge^2 V}$ even at the pinch point $(\xi, u, v) = (0,0,0)$.

The zero section $\sigma \hookrightarrow Z$ is now replaced by $\sigma^* := \nu_Z^*(\sigma) \hookrightarrow \tilde{Z}$. It consists of two irreducible components. One of them does not appear in the first patch, because it is specified by $\tilde{\xi} = 1/\tilde{u} = 0$. The other component is the exceptional locus E of this blow-up. It is given by $\xi = 0$. If the covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$ is defined by $\tilde{\bar{c}}_{\wedge^2 V} := \sigma^* \cdot \widetilde{C}_{\wedge^2 V}$, then it is specified by

$$\tilde{u}^2 = v, \qquad \xi = 0. \tag{399}$$

$\tilde{u}$ can be chosen as a local coordinate of the covering matter curve $\tilde{\bar{c}}_{\wedge^2 V}$, while v is the local coordinate of the matter curve $\bar{c}_{\wedge^2 V}$. The map $\nu_{\bar{c}_{\wedge^2 V}} : \tilde{\bar{c}}_{\wedge^2 V} \to \bar{c}_{\wedge^2 V}$ is clearly a degree-2 cover, $v = \tilde{u}^2$, and each type (c) point on $\bar{c}_{\wedge^2 V}$ $(v = 0)$ is a branch point of this degree-2 cover.

References

[1] R. Tatar and T. Watari, "Proton decay, Yukawa couplings and underlying gauge symmetry in string theory," Nucl. Phys. B **747**, 212 (2006) [arXiv:hep-th/0602238].

[2] E. Witten, "Strong Coupling Expansion Of Calabi-Yau Compactification," Nucl. Phys. B **471**, 135 (1996) [arXiv:hep-th/9602070].

[3] P. Horava and E. Witten, "Heterotic and type I string dynamics from eleven dimensions," Nucl. Phys. B **460**, 506 (1996) [arXiv:hep-th/9510209].

[4] C. Vafa, "Evidence for F-Theory," Nucl. Phys. B **469**, 403 (1996) [arXiv:hep-th/9602022].

[5] M. Grana, "Flux compactifications in string theory: A comprehensive review," Phys. Rept. **423**, 91 (2006) [arXiv:hep-th/0509003]; M. R. Douglas and S. Kachru, "Flux compactification," Rev. Mod. Phys. **79**, 733 (2007) [arXiv:hep-th/0610102]; F. Denef, "Les Houches Lectures on Constructing String Vacua," arXiv:0803.1194 [hep-th].

[6] D. R. Morrison and C. Vafa, "Compactifications of F-Theory on Calabi–Yau Threefolds – I," Nucl. Phys. B **473**, 74 (1996) [arXiv:hep-th/9602114].

[7] D. R. Morrison and C. Vafa, "Compactifications of F-Theory on Calabi–Yau Threefolds – II," Nucl. Phys. B **476**, 437 (1996) [arXiv:hep-th/9603161].

[8] M. Berkooz, M. R. Douglas and R. G. Leigh, "Branes intersecting at angles," Nucl. Phys. B **480**, 265 (1996) [arXiv:hep-th/9606139].

[9] B. Acharya and E. Witten, "Chiral fermions from manifolds of G(2) holonomy," arXiv:hep-th/0109152.

[10] R. Donagi and M. Wijnholt, "Model Building with F-Theory," arXiv:0802.2969 [hep-th].

[11] C. Beasley, J. J. Heckman and C. Vafa, "GUTs and Exceptional Branes in F-theory - I," arXiv:0802.3391 [hep-th].

[12] R. Friedman, J. Morgan and E. Witten, "Vector bundles and F theory," Commun. Math. Phys. **187** (1997) 679 [arXiv:hep-th/9701162].

[13] M. Bershadsky, K. A. Intriligator, S. Kachru, D. R. Morrison, V. Sadov and C. Vafa, "Geometric singularities and enhanced gauge symmetries," Nucl. Phys. B **481**, 215 (1996) [arXiv:hep-th/9605200].

[14] M. Bershadsky, A. Johansen, T. Pantev and V. Sadov, "On four-dimensional compactifications of F-theory," Nucl. Phys. B **505**, 165 (1997) [arXiv:hep-th/9701165].

[15] G. Curio and R. Y. Donagi, "Moduli in N = 1 heterotic/F-theory duality," Nucl. Phys. B **518** (1998) 603 [arXiv:hep-th/9801057].

[16] G. Curio, "Chiral matter and transitions in heterotic string models," Phys. Lett. B **435**, 39 (1998) [arXiv:hep-th/9803224].

[17] D. E. Diaconescu and G. Ionesei, "Spectral covers, charged matter and bundle cohomology," JHEP **9812**, 001 (1998) [arXiv:hep-th/9811129].

[18] R. Donagi, Y. H. He, B. A. Ovrut and R. Reinbacher, "The particle spectrum of heterotic compactifications," JHEP **0412**, 054 (2004) [arXiv:hep-th/0405014].

[19] R. Blumenhagen, S. Moster, R. Reinbacher and T. Weigand, "Massless spectra of three generation U(N) heterotic string vacua," JHEP **0705**, 041 (2007) [arXiv:hep-th/0612039].

[20] R. Donagi, "Taniguchi lecture on principal bundles on elliptic fibrations," arXiv:hep-th/9802094.

[21] R. Friedman, J. W. Morgan and E. Witten, "Principal G-bundles over elliptic curves," Math. Res. Lett. **5**, 97 (1998) [arXiv:alg-geom/9707004].

[22] B. A. Ovrut, T. Pantev and J. Park, "Small instanton transitions in heterotic M-theory," JHEP **0005**, 045 (2000) [arXiv:hep-th/0001133].

[23] R. Tatar and T. Watari, "A stable proton without R-parity: Implications for the LSP," Phys. Lett. B **646**, 258 (2007) [arXiv:hep-ph/0605315].

[24] M. Kuriyama, H. Nakajima and T. Watari, "A Theoretical Framework for R-parity Violation," arXiv:0802.2584 [hep-ph].

[25] B. Andreas and G. Curio, "Three-branes and five-branes in N = 1 dual string pairs," Phys. Lett. B **417**, 41 (1998) [arXiv:hep-th/9706093].

[26] M. Atiyah, "Vector Bundles over an Elliptic Curve," Proc. London Math. Soc. vol. 7, (1957), 414-452.

[27] R. Hartshorne, "*Algebraic Geometry,*" Springer (1977).

[28] P. Griffiths and J. Harris, "*Principles of algebraic geometry,*" Wiley (1994)

[29] Yu. I. Manin, "*Cubic Forms*," North-Holland (1974).

[30] S. Katz, P. Mayr and C. Vafa, "Mirror symmetry and exact solution of 4D N = 2 gauge theories. I," Adv. Theor. Math. Phys. **1**, 53 (1998) [arXiv:hep-th/9706110];
P. Berglund and P. Mayr, "Heterotic string/F-theory duality from mirror symmetry," Adv. Theor. Math. Phys. **2**, 1307 (1999) [arXiv:hep-th/9811217].

[31] K. Dasgupta and S. Mukhi, "A note on low-dimensional string compactifications," Phys. Lett. B **398**, 285 (1997) [arXiv:hep-th/9612188].

[32] B. Andreas and G. Curio, "On discrete twist and four-flux in N = 1 heterotic/F-theory compactifications," Adv. Theor. Math. Phys. **3**, 1325 (1999) [arXiv:hep-th/9908193].

[33] E. Witten, "On flux quantization in M-theory and the effective action," J. Geom. Phys. **22**, 1 (1997) [arXiv:hep-th/9609122].

[34] R. Blumenhagen, G. Honecker and T. Weigand, "Loop-corrected compactifications of the heterotic string with line bundles," JHEP **0506** (2005) 020 [arXiv:hep-th/0504232].

[35] E. Witten, "Symmetry Breaking Patterns In Superstring Models," Nucl. Phys. B **258**, 75 (1985).

[36] H. Jockers and J. Louis, "The effective action of D7-branes in N = 1 Calabi-Yau orientifolds," Nucl. Phys. B **705**, 167 (2005) [arXiv:hep-th/0409098].

[37] D. S. Freed and E. Witten, "Anomalies in string theory with D-branes," arXiv:hep-th/9907189.

[38] I. Brunner and K. Hori, "Orientifolds and mirror symmetry," JHEP **0411**, 005 (2004) [arXiv:hep-th/0303135];
I. Brunner, K. Hori, K. Hosomichi and J. Walcher, "Orientifolds of Gepner models," JHEP **0702**, 001 (2007) [arXiv:hep-th/0401137].

[39] T. Watari and T. Yanagida, "Product-group unification in type IIB string thoery," Phys. Rev. D **70**, 036009 (2004) [arXiv:hep-ph/0402160]. (See version 2 on the arXiv.)

[40] R. Minasian and G. W. Moore, "K-theory and Ramond-Ramond charge," JHEP **9711**, 002 (1997) [arXiv:hep-th/9710230].

[41] R. Blumenhagen, V. Braun, B. Kors and D. Lust, "Orientifolds of K3 and Calabi-Yau manifolds with intersecting D-branes," JHEP **0207**, 026 (2002) [arXiv:hep-th/0206038].

[42] I. Brunner, M. R. Douglas, A. E. Lawrence and C. Romelsberger, "D-branes on the quintic," JHEP **0008**, 015 (2000) [arXiv:hep-th/9906200].

[43] Y. K. Cheung and Z. Yin, "Anomalies, branes, and currents," Nucl. Phys. B **517**, 69 (1998) [arXiv:hep-th/9710206].

[44] J. F. Morales, C. A. Scrucca and M. Serone, "Anomalous couplings for D-branes and O-planes," Nucl. Phys. B **552**, 291 (1999) [arXiv:hep-th/9812071];
B. J. Stefanski, "Gravitational couplings of D-branes and O-planes," Nucl. Phys. B **548**,

275 (1999) [arXiv:hep-th/9812088];
C. A. Scrucca and M. Serone, "Anomalies and inflow on D-branes and O-planes," Nucl. Phys. B **556**, 197 (1999) [arXiv:hep-th/9903145].

[45] P. Candelas, D. E. Diaconescu, B. Florea, D. R. Morrison and G. Rajesh, "Codimension-three bundle singularities in F-theory," JHEP **0206**, 014 (2002) [arXiv:hep-th/0009228].

[46] G. Rajesh, "Toric geometry and F-theory/heterotic duality in four dimensions," JHEP **9812**, 018 (1998) [arXiv:hep-th/9811240].

[47] G. Aldazabal, L. E. Ibanez, F. Quevedo and A. M. Uranga, "D-branes at singularities: A bottom-up approach to the string embedding of the standard model," JHEP **0008**, 002 (2000) [arXiv:hep-th/0005067].

[48] A. M. Uranga, "Local models for intersecting brane worlds," JHEP **0212**, 058 (2002) [arXiv:hep-th/0208014].

[49] E. Witten, "Deconstruction, G(2) holonomy, and doublet-triplet splitting," arXiv:hep-ph/0201018.

[50] I. Affleck, M. Dine and N. Seiberg, "Dynamical Supersymmetry Breaking In Four-Dimensions And Its Phenomenological Implications," Nucl. Phys. B **256**, 557 (1985).

[51] K. Dasgupta, G. Rajesh and S. Sethi, "M theory, orientifolds and G-flux," JHEP **9908**, 023 (1999) [arXiv:hep-th/9908088].

[52] H. Hayashi, T. Kawano, R. Tatar and T. Watari, "Codimension-3 Singularities and Yukawa Couplings in F-theory," arXiv:0901.4941 [hep-th].

[53] R. Donagi and M. Wijnholt, "Higgs Bundles and UV Completion in F-Theory," arXiv:0904.1218 [hep-th].